EE101 W19 Lecture 18, May 7, 2019

HW #9 for Quiz 9 on May 12

[1] 6.29
[2] 6.35
[3] 6.38
[4] 6.50
[5] 6.54
[6] 7.1
[7] 7.10 (a), (b), (e)
[8] 7.22
[9] 7.36
[10] 7.86

Quiz 8 Average = 9.00
σ = 1.53

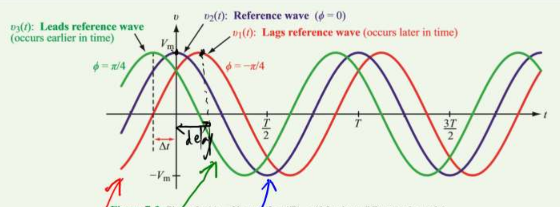


Figure 7-2: Plots of $v(t) = V_m \cos[(2\pi t/T) + \phi]$ for three different values of ϕ.

$v(t) = V_m \cos\omega t \qquad \mathbb{V} = V_m e^{j0^\circ} = V_m \angle 0^\circ$

$v(t) = V_m \cos(\omega t + \pi/4) \qquad \mathbb{V} = V_m e^{j\pi/4} = V_m \angle 45^\circ$

$v(t) = V_m \cos(\omega t - \pi/4) \qquad \mathbb{V} = V_m e^{-j\pi/4} = V_m \angle -45^\circ$

$\frac{d}{dt} f(t) \longleftrightarrow sF(s)$

time derivative, and vice versa. We surmise from Eq. (7.36) that

$$\frac{di}{dt} \leftrightarrow j\omega \mathbf{I}, \qquad (7.37)$$

or:

▶ Differentiation of a time function $i(t)$ in the time domain is equivalent to multiplication of its phasor counterpart $\mathbf{I}$ by $j\omega$ in the phasor domain. ◀

Similarly,

$$\int i\,dt = \int \Re\mathfrak{e}[\mathbf{I}e^{j\omega t}]\,dt = \Re\mathfrak{e}\left[\int \mathbf{I}e^{j\omega t}\,dt\right] = \Re\mathfrak{e}\left[\underbrace{\frac{\mathbf{I}}{j\omega}}_{\text{phasor of } \int i\,dt} e^{j\omega t}\right]. \qquad (7.38)$$

or

$$\int i\,dt \leftrightarrow \frac{\mathbf{I}}{j\omega}. \qquad (7.39)$$

Table 7-3: Time-domain sinusoidal functions $x(t)$ and their cosine-reference phasor-domain counterparts $\mathbf{X}$, where $x(t) = \Re\mathfrak{e}\,[\mathbf{X}e^{j\omega t}]$.

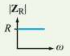
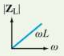
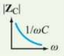
$x(t)$		$\mathbf{X}$
$A\cos\omega t$	↔	A
$A\cos(\omega t + \phi)$	↔	$Ae^{j\phi}$
$-A\cos(\omega t + \phi)$	↔	$Ae^{j(\phi \pm \pi)}$
$A\sin\omega t$	↔	$Ae^{-j\pi/2} = -jA$
$A\sin(\omega t + \phi)$	↔	$Ae^{j(\phi - \pi/2)}$
$-A\sin(\omega t + \phi)$	↔	$Ae^{j(\phi + \pi/2)}$
$\frac{d}{dt}(x(t))$	↔	$j\omega\mathbf{X}$
$\frac{d}{dt}[A\cos(\omega t + \phi)]$	↔	$j\omega Ae^{j\phi}$
$\int x(t)\,dt$	↔	$\frac{1}{j\omega}\mathbf{X}$
$\int A\cos(\omega t + \phi)\,dt$	↔	$\frac{1}{j\omega}Ae^{j\phi}$

$$i_R = \Re\mathfrak{e}[\mathbf{I}_R e^{j\omega t}]. \qquad (7.41b)$$

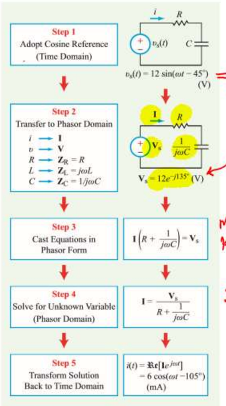
Table 7-3: Time-domain sinusoidal functions $x(t)$ and their cosine-reference phasor-domain counterparts $\mathbf{X}$, where $x(t) = \Re\mathfrak{e}\,[\mathbf{X}e^{j\omega t}]$.

$x(t)$		$\mathbf{X}$
$A\cos\omega t$	↔	A
$A\cos(\omega t + \phi)$	↔	$Ae^{j\phi}$
$-A\cos(\omega t + \phi)$	↔	$Ae^{j(\phi \pm \pi)}$
$A\sin\omega t$	↔	$Ae^{-j\pi/2} = -jA$
$A\sin(\omega t + \phi)$	↔	$Ae^{j(\phi - \pi/2)}$
$-A\sin(\omega t + \phi)$	↔	$Ae^{j(\phi + \pi/2)}$
$\frac{d}{dt}(x(t))$	↔	$j\omega\mathbf{X}$
$\frac{d}{dt}[A\cos(\omega t + \phi)]$	↔	$j\omega Ae^{j\phi}$
$\int x(t)\,dt$	↔	$\frac{1}{j\omega}\mathbf{X}$
$\int A\cos(\omega t + \phi)\,dt$	↔	$\frac{1}{j\omega}Ae^{j\phi}$

Table 7-4: Summary of v–i properties for R, L, and C.

Property	R	L	C
v–i	$v = Ri$	$v = L\frac{di}{dt}$	$i = C\frac{dv}{dt}$
$\mathbf{V}$–$\mathbf{I}$	$\mathbf{V} = R\mathbf{I}$	$\mathbf{V} = j\omega L\mathbf{I}$	$\mathbf{V} = \frac{\mathbf{I}}{j\omega C}$
$\mathbf{Z}$	R	$j\omega L$	$\frac{1}{j\omega C}$
dc equivalent	R	Short circuit	Open circuit
High-frequency equivalent	R	Open circuit	Short circuit
Frequency response			

Phasor domain circuit analysis



Step 1
Adopt Cosine Reference
(Time Domain)

$v_s(t) = 12\sin(\omega t - 45^\circ)$ (V) $= 12\cos(\omega t - 45^\circ - 90^\circ)$
$= 12\cos(\omega t - 135^\circ)$

Step 2
Transfer to Phasor Domain
$i \rightarrow \mathbf{I}$
$v \rightarrow \mathbf{V}$
$R \rightarrow \mathbf{Z}_R = R$
$L \rightarrow \mathbf{Z}_L = j\omega L$
$C \rightarrow \mathbf{Z}_C = 1/j\omega C$

$\mathbf{V}_s = 12e^{-j135^\circ}$ (V)

Step 3
Cast Equations in Phasor Form

$\mathbf{I}\left(R + \frac{1}{j\omega C}\right) = \mathbf{V}_s$

Mesh analysis
KVL

Step 4
Solve for Unknown Variable
(Phasor Domain)

$\mathbf{I} = \frac{\mathbf{V}_s}{R + \frac{1}{j\omega C}}$

$\mathbb{I} = \mathbb{V}_s / Z$
$Z = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}\; e^{j\tan^{-1}\left(\frac{-\frac{1}{\omega C}}{R}\right)}$

Step 5
Transform Solution
Back to Time Domain

$i(t) = \Re\mathfrak{e}[\mathbf{I}e^{j\omega t}]$
$= 6\cos(\omega t - 105^\circ)$
(mA)

Figure 7-7: Five-step procedure for analyzing ac circuits using the phasor-domain technique.

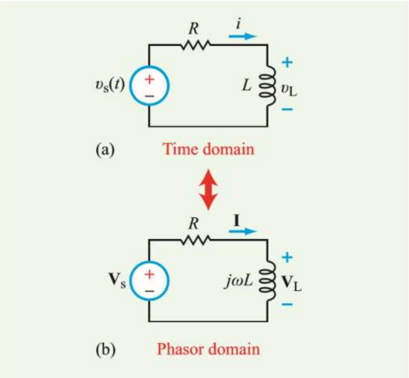


Figure 7-8: RL circuit of Example 7-5.

$$\frac{1}{j\omega C} = \frac{-j^2}{j\omega C} = -j\frac{1}{\omega C}$$

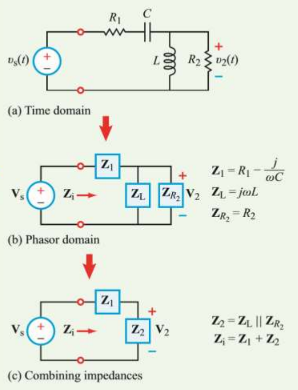


Figure 7-12: Circuit for Example 7-6.

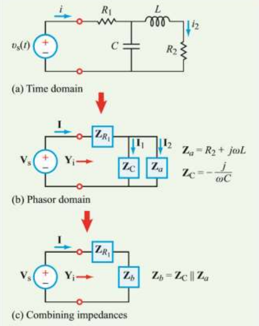


Figure 7-13: Circuit for Example 7-7.

Determine (a) the input admittance $\mathbf{Y}_i$, given that $R_1 = 10\ \Omega$, $R_2 = 30\ \Omega$, $L = 2\ \mu$H, and $C = 10$ nF, and (b) the current $i_2(t)$ flowing through R_2.

Solution: **(a)** We start by converting $v_s(t)$ to cosine format:

$$\begin{aligned} v_s(t) &= 4\sin(10^7 t + 15^\circ) \\ &= 4\cos(10^7 t + 15^\circ - 90^\circ) = 4\cos(10^7 t - 75^\circ)\ \text{V}. \end{aligned}$$

The corresponding phasor voltage is

$$\mathbf{V}_s = 4e^{-j75^\circ}\ \text{V},$$

and the impedances shown in **Fig. 7-13(b)** are given by

$$\mathbf{Z}_{R_1} = R_1 = 10\ \Omega,$$

$$\mathbf{Z}_C = \frac{-j}{\omega C} = \frac{-j}{10^7 \times 10^{-8}} = -j10\ \Omega,$$

$$\mathbf{Z}_a = R_2 + j\omega L = 30 + j10^7 \times 2 \times 10^{-6} = (30 + j20)\ \Omega.$$

In **Fig. 7-13(c)**, $\mathbf{Z}_b$ represents the parallel combination of $\mathbf{Z}_C$ and $\mathbf{Z}_a$.

$$Z_C \parallel Z_a = \frac{Z_C\, Z_a}{Z_C + Z_a}$$

$$\begin{aligned} \mathbf{Z}_b &= \mathbf{Z}_C \parallel \mathbf{Z}_a \\ &= \frac{(-j10)(30 + j20)}{-j10 + 30 + j20} \\ &= \frac{20 - j30}{3 + j1} = \frac{(20 - j30)}{(3 + j1)}\frac{(3 - j1)}{(3 - j1)} = (3 - j11)\ \Omega. \end{aligned}$$

$$R_C \parallel R_a = \frac{R_C \cdot R_a}{R_C + R_C}$$

The input impedance is

$$\mathbf{Z}_i = \mathbf{Z}_{R_1} + \mathbf{Z}_b = 10 + 3 - j11 = (13 - j11)\ \Omega,$$

and its reciprocal is

$$\begin{aligned} \mathbf{Y}_i = \frac{1}{\mathbf{Z}_i} &= \frac{1}{13 - j11} \times \frac{13 + j11}{13 + j11} \\ &= \frac{13 + j11}{169 + 121} \\ &= (4.5 + j3.8) \times 10^{-2} = 5.89 \times 10^{-2} e^{-j40.2^\circ}\ \text{S}. \end{aligned}$$

(b) The current **I** is given by

$$\mathbf{I} = \mathbf{V}_s\mathbf{Y}_i = (4e^{-j75^\circ})(5.89\times 10^{-2} e^{-j40.2^\circ}) = 0.235e^{-j34.8^\circ}\ \text{A}.$$

By current division in **Fig. 7-13(b)**,

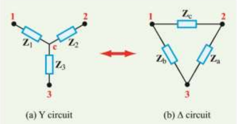
$$\begin{aligned} \mathbf{I}_2 &= \frac{\mathbf{Z}_C}{\mathbf{Z}_a + \mathbf{Z}_C}\mathbf{I} \\ &= \frac{-j10}{30 + j20 - j10} \times 0.235e^{-j34.8^\circ} \\ &= \frac{2.35e^{-j34.8^\circ} \cdot e^{-j90^\circ}}{31.6e^{j18.4^\circ}} = 7.4 \times 10^{-2} e^{-j143.2^\circ}\ \text{A}. \end{aligned}$$

The corresponding current in the time domain is

$$\begin{aligned} i_2(t) &= \mathfrak{Re}[\mathbf{I}_2 e^{j\omega t}] = \mathfrak{Re}[7.4 \times 10^{-2} e^{-j143.2^\circ} e^{j10^7 t}] \\ &= 7.4 \times 10^{-2}\cos(10^7 t - 143.2^\circ)\ \text{A}. \end{aligned}$$

$$\cos(\theta - 90^\circ) = \cos\theta\cos 90^\circ + \sin\theta\sin 90^\circ = \sin\theta$$

$$\cos(\theta + 90^\circ) = \cos\theta\cos 90^\circ - \sin\theta\sin 90^\circ = \ominus\sin\theta$$



(a) Y circuit (b) Δ circuit

Figure 7-14: Y–Δ equivalent circuits.

$$\mathbf{Z}_3 = \frac{\mathbf{Z}_a\mathbf{Z}_b}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}. \qquad (7.79c)$$

Y→ Δ transformation:

$$\mathbf{Z}_a = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_1\mathbf{Z}_3}{\mathbf{Z}_1}, \qquad (7.80a)$$

$$\mathbf{Z}_b = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_1\mathbf{Z}_3}{\mathbf{Z}_2}, \qquad (7.80b)$$

$$\mathbf{Z}_c = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_1\mathbf{Z}_3}{\mathbf{Z}_3}. \qquad (7.80c)$$

Δ →Y transformation:

$$\mathbf{Z}_1 = \frac{\mathbf{Z}_b\mathbf{Z}_c}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}, \qquad (7.79a)$$

$$\mathbf{Z}_2 = \frac{\mathbf{Z}_a\mathbf{Z}_c}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}, \qquad (7.79b)$$

$$\mathbf{Z}_3 = \frac{\mathbf{Z}_a\mathbf{Z}_b}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}. \qquad (7.79c)$$

Δ →Y transformation:

$$\mathbf{Z}_3 = \frac{\mathbf{Z}_a\mathbf{Z}_b}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}. \qquad (7.79c)$$

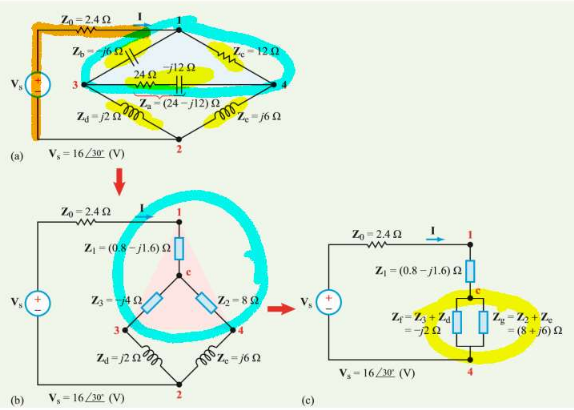
$$\mathbf{Z}_2 = \frac{\mathbf{Z}_a\mathbf{Z}_c}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}, \qquad (7.79b)$$

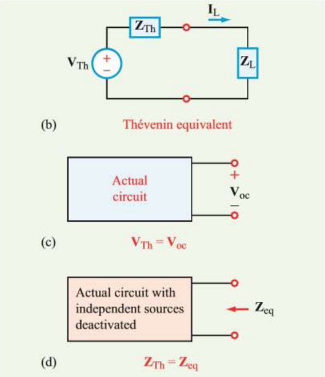
$$\begin{aligned}\mathbf{Z}_1 &= \frac{\mathbf{Z}_b\mathbf{Z}_c}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}\\ &= \frac{-j6 \times 12}{24 - j12 - j6 + 12} = \frac{-j72}{36 - j18} = (0.8 - j1.6)\ \Omega,\\ \mathbf{Z}_2 &= \frac{\mathbf{Z}_a\mathbf{Z}_c}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c} = \frac{(24 - j12) \times 12}{36 - j18} = 8\ \Omega,\end{aligned}$$

and

$$\mathbf{Z}_3 = \frac{\mathbf{Z}_b\mathbf{Z}_a}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c} = \frac{-j6(24 - j12)}{36 - j18} = -j4\ \Omega.$$

In **Fig. 7-15(c)**, $\mathbf{Z}_f$ represents the series combination of $\mathbf{Z}_3$ and $\mathbf{Z}_d$,



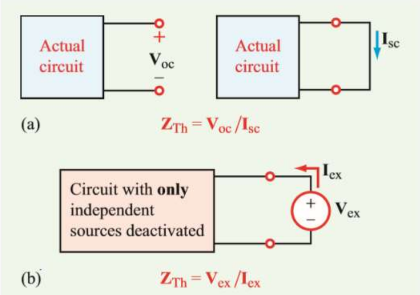


(b) Thévenin equivalent

(c) $\mathbf{V}_{Th} = \mathbf{V}_{oc}$

(d) $\mathbf{Z}_{Th} = \mathbf{Z}_{eq}$

Figure 7-17: Thévenin-equivalent method for a circuit with no dependent sources.



(a) $\mathbf{Z}_{Th} = \mathbf{V}_{oc}/\mathbf{I}_{sc}$

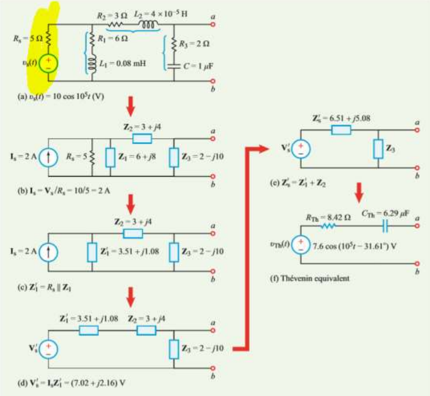
(b) $\mathbf{Z}_{Th} = \mathbf{V}_{ex}/\mathbf{I}_{ex}$

Figure 7-18: The (a) open-circuit/short-circuit method and (b) the external-source method are both suitable for determining $\mathbf{Z}_{Th}$, whether or not the circuit contains dependent sources.

Open-circuit / short-circuit method

$$\mathbf{Z}_{Th} = \frac{\mathbf{V}_{oc}}{\mathbf{I}_{sc}}, \qquad (7.83)$$

where $\mathbf{I}_{sc}$ is the short-circuit current at the circuit's output terminals (**Fig. 7-18(a)**).



(a) $v_s(t) = 10 \cos 10^5 t$ (V)

(b) $\mathbf{I}_s = \mathbf{V}_s/R_s = 10/5 = 2$ A

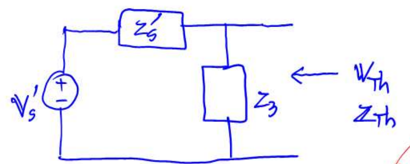
(c) $\mathbf{Z}_1' = R_s \parallel \mathbf{Z}_1$

(d) $\mathbf{V}_s' = \mathbf{I}_s\mathbf{Z}_1' = (7.02 + j2.16)$ V

(e) $\mathbf{Z}_s' = \mathbf{Z}_1' + \mathbf{Z}_2$

(f) Thévenin equivalent

Figure 7-19: Using source transformation to simplify the circuit of Example 7-9. (All impedances are in ohms.)



$$V_{Th} = V_s' \frac{Z_3}{Z_3 + Z_s'}$$

$$Z_{Th} = Z_3 \parallel Z_s'$$

$$V_{Th} = V_s' \frac{2-j10}{(2-j10)+(6.51+j5.08)} = V_s' \frac{2-j10}{8.51-j4.92}$$

$$\simeq (7.02+j2.16) \frac{2-j10}{8.51-j4.92}$$

$$V_{th} e^{j\theta_{th}} \ ?$$

complex number coordinate

$$(a+jb)\frac{(c+jd)}{(e+jf)}$$

$$\frac{(a+jb)(c+jd)(e-jf)}{(e+jf)(e-jf)} = \frac{\cdots}{(e^2+f^2)}$$

$$= \alpha + j\beta \Rightarrow \sqrt{\alpha^2+\beta^2}\angle \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

polar

$a+jb = \sqrt{a^2+b^2}\,\angle \tan^{-1}\frac{b}{a} = M_1 e^{j\theta_1}$

$c+jd = \sqrt{c^2+d^2}\,\angle \tan^{-1}\frac{d}{c} = M_2 e^{j\theta_2}$

$e+jf = \sqrt{e^2+f^2}\,\angle \tan^{-1}\frac{f}{e} = M_3 e^{j\theta_3}$

$\Rightarrow \frac{M_1 \cdot M_2}{M_3} e^{j(\theta_1+\theta_2-\theta_3)}$

Exercise 7-15: Write down the node-voltage matrix equation for the circuit in **Fig. E7.15.**

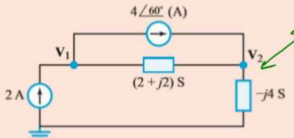


Figure E7.15

Answer:

$$\begin{bmatrix} (2+j2) & -(2+j2) \\ -(2+j2) & (2-j2) \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 2-4e^{j60^\circ} \\ 4e^{j60^\circ} \end{bmatrix}.$$

(See CAD)

KCL

$4\angle 60^\circ$

$= V_2(-j4)$

$+(V_2-V_1)(2+j2)$

$= -V_1(2+j2)$

$+V_2\underbrace{(2+j2-j4)}_{2-j2}$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$B = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$ $\quad B^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$

$BB^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} (2)(3)+4(-1) & 2(-4)+4(2) \\ 1(3)+3(-1) & 1(-4)+3(2) \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

$|B| = (2)(3) - (4)(1)$

$|B| = 6 - 4 = 2$

$\begin{bmatrix} 2+j2 & -(2+j2) \\ -(2+j2) & (2-j2) \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 2-4e^{+j60^\circ} \\ 4e^{+j60^\circ} \end{bmatrix}$

$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2+j2 & -(2+j2) \\ -(2+j2) & (2-j2) \end{bmatrix}}_{A}^{-1}\begin{bmatrix} 2-4e^{+j60^\circ} \\ 4e^{+j60^\circ} \end{bmatrix}$

$\det A = (2+j2)(2-j2) - (-(2+j2))(-(2+j2))$

$= 2(2)+j(-j)2(2) - (2+j2)(2+j2)$

$= 4-(-1)4-(4+j4+j4+j^2 4) = 8-(0+j8) = \underline{8-j8}$

$A^{-1} = \frac{1}{8-j8}\begin{bmatrix} (2-j2) & (2+j2) \\ (2+j2) & (2+j2) \end{bmatrix}$

$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = A^{-1}\begin{bmatrix} 2-4e^{+j60^\circ} \\ 4e^{+j40^\circ} \end{bmatrix} = \frac{1}{8-j8}\begin{bmatrix} (2-j2) & (2+j2) \\ (2+j2) & (2+j2) \end{bmatrix}\begin{bmatrix} 2-4(\cos 60^\circ + j\sin 60^\circ) \\ = 2-2-j2\sqrt{3} = -j2\sqrt{3} \\ 2+j2\sqrt{3} \end{bmatrix}$

$= \frac{1}{8-j8}\begin{bmatrix} (2-j2)(-j2\sqrt{3})+(2+j2)(2+j2\sqrt{3}) \\ (2+j2)(-j2\sqrt{3})+(2+j2)(2+j2\sqrt{3}) \end{bmatrix}$

$= \frac{1}{8-j8}\begin{bmatrix} -j4\sqrt{3}-4\sqrt{3}+4+j4\sqrt{3}+j4-4\sqrt{3} \\ =\underline{4-8\sqrt{3}+j4} \\ j4\sqrt{3}+4\sqrt{3}+4+j4\sqrt{3}+j4-4\sqrt{3} \\ =\underline{4+j4} \end{bmatrix}$

$= \begin{bmatrix} \frac{(4-8\sqrt{3})+j4}{8-j8} \\ \frac{4+j4}{8-j8} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{(1-2\sqrt{3})+j1}{2-j2} \\ \frac{1+j1}{1-j1} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{((1-2\sqrt{3})+j1)(2+j2)}{8} \\ \frac{(1+j1)(1+j1)}{\underbrace{(1-j1)(1+j1)}_{=2}} \end{bmatrix}$

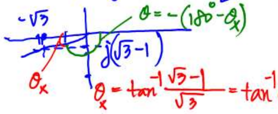
$(\sqrt{2}e^{j45^\circ})^2 = 2e^{+j90^\circ}$

$j =$

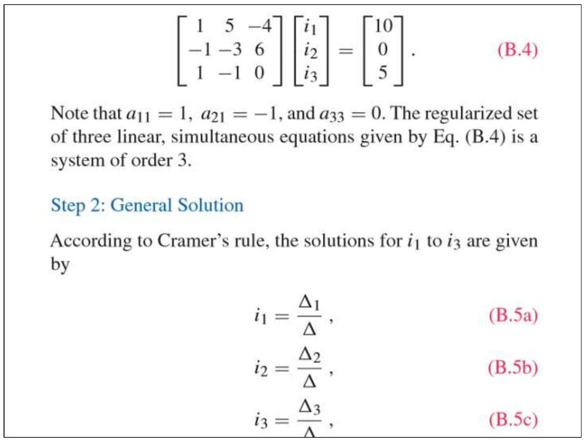
$= \frac{1}{2}\begin{bmatrix} \frac{((1-2\sqrt{3})+j1)(\cancel{2}^{1}+j\cancel{2}^{1})}{\cancel{8}\,4} = \frac{\cancel{2}^{1}(-\sqrt{3}+j(1-\sqrt{3}))}{\cancel{4}\,2} \\ j \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{4}(-\sqrt{3}-j(\sqrt{3}-1)) \\ \frac{1}{2}j \end{bmatrix} = \begin{bmatrix} \overset{0.47}{\frac{1}{4}(1.88)}\, e^{j(-157.1^\circ)} \\ \frac{1}{2}e^{j90^\circ} \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

ans



$\theta_x = \tan^{-1}\frac{\sqrt{3}-1}{\sqrt{3}} = \tan^{-1}0.423 = 0.4\text{ rad} = 22.9^\circ$



$$\begin{bmatrix} 1 & 5 & -4 \\ -1 & -3 & 6 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 5 \end{bmatrix}. \qquad \text{(B.4)}$$

Note that $a_{11} = 1$, $a_{21} = -1$, and $a_{33} = 0$. The regularized set of three linear, simultaneous equations given by Eq. (B.4) is a system of order 3.

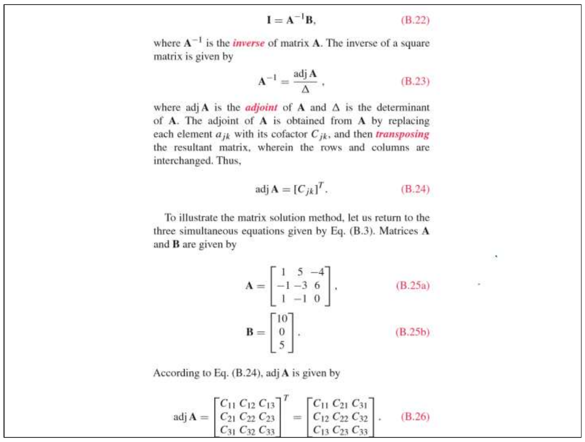
Step 2: General Solution

According to Cramer's rule, the solutions for i_1 to i_3 are given by

$$i_1 = \frac{\Delta_1}{\Delta}, \qquad \text{(B.5a)}$$

$$i_2 = \frac{\Delta_2}{\Delta}, \qquad \text{(B.5b)}$$

$$i_3 = \frac{\Delta_3}{\Delta}, \qquad \text{(B.5c)}$$



$$\mathbf{I} = \mathbf{A}^{-1}\mathbf{B}, \qquad \text{(B.22)}$$

where $\mathbf{A}^{-1}$ is the ***inverse*** of matrix $\mathbf{A}$. The inverse of a square matrix is given by

$$\mathbf{A}^{-1} = \frac{\operatorname{adj} \mathbf{A}}{\Delta}, \qquad \text{(B.23)}$$

where $\operatorname{adj} \mathbf{A}$ is the ***adjoint*** of $\mathbf{A}$ and Δ is the determinant of $\mathbf{A}$. The adjoint of $\mathbf{A}$ is obtained from $\mathbf{A}$ by replacing each element a_{jk} with its cofactor C_{jk}, and then ***transposing*** the resultant matrix, wherein the rows and columns are interchanged. Thus,

$$\operatorname{adj} \mathbf{A} = [C_{jk}]^T. \qquad \text{(B.24)}$$

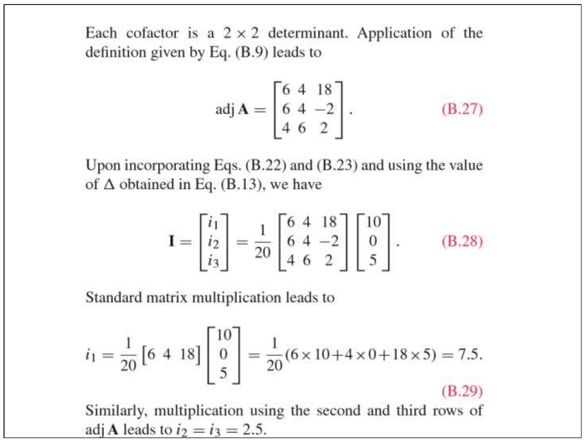
To illustrate the matrix solution method, let us return to the three simultaneous equations given by Eq. (B.3). Matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & -4 \\ -1 & -3 & 6 \\ 1 & -1 & 0 \end{bmatrix}, \qquad \text{(B.25a)}$$

$$\mathbf{B} = \begin{bmatrix} 10 \\ 0 \\ 5 \end{bmatrix}. \qquad \text{(B.25b)}$$

According to Eq. (B.24), $\operatorname{adj} \mathbf{A}$ is given by

$$\operatorname{adj} \mathbf{A} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}. \qquad \text{(B.26)}$$



Each cofactor is a 2×2 determinant. Application of the definition given by Eq. (B.9) leads to

$$\operatorname{adj} \mathbf{A} = \begin{bmatrix} 6 & 4 & 18 \\ 6 & 4 & -2 \\ 4 & 6 & 2 \end{bmatrix}. \qquad \text{(B.27)}$$

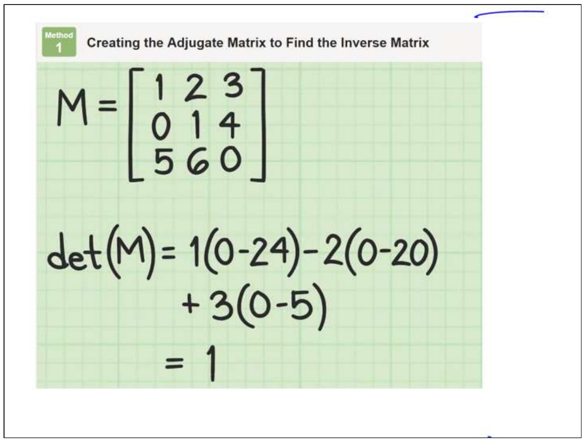
Upon incorporating Eqs. (B.22) and (B.23) and using the value of Δ obtained in Eq. (B.13), we have

$$\mathbf{I} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 6 & 4 & 18 \\ 6 & 4 & -2 \\ 4 & 6 & 2 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \\ 5 \end{bmatrix}. \qquad \text{(B.28)}$$

Standard matrix multiplication leads to

$$i_1 = \frac{1}{20} [6 \;\; 4 \;\; 18] \begin{bmatrix} 10 \\ 0 \\ 5 \end{bmatrix} = \frac{1}{20}(6 \times 10 + 4 \times 0 + 18 \times 5) = 7.5. \qquad \text{(B.29)}$$

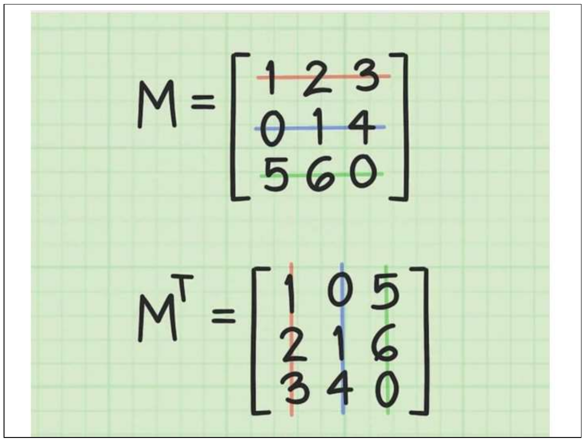
Similarly, multiplication using the second and third rows of $\operatorname{adj} \mathbf{A}$ leads to $i_2 = i_3 = 2.5$.



Method 1 Creating the Adjugate Matrix to Find the Inverse Matrix

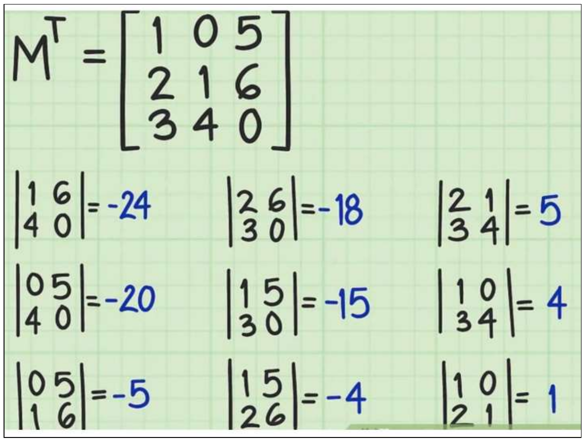
$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

$$\det(M) = 1(0-24) - 2(0-20)$$
$$+ 3(0-5)$$
$$= 1$$



$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

$$M^T = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$$



$$M^T = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$$

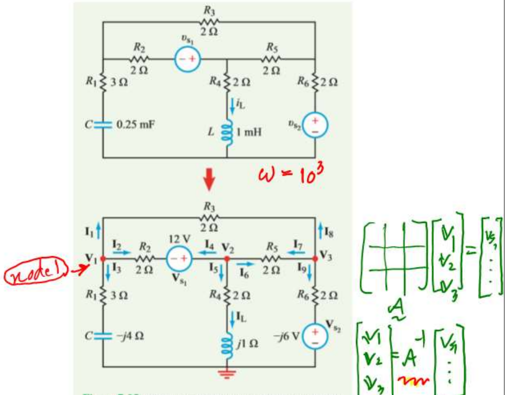
$$\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} = -24 \qquad \begin{vmatrix} 2 & 6 \\ 3 & 0 \end{vmatrix} = -18 \qquad \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 5$$

$$\begin{vmatrix} 0 & 5 \\ 4 & 0 \end{vmatrix} = -20 \qquad \begin{vmatrix} 1 & 5 \\ 3 & 0 \end{vmatrix} = -15 \qquad \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = 4$$

$$\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = -5 \qquad \begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix} = -4 \qquad \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1$$

$$\mathrm{Adj}(M) = \begin{bmatrix} -24 & -18 & 5 \\ -20 & -15 & 4 \\ -5 & -4 & 1 \end{bmatrix} \times \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\mathrm{Adj}(M) = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$$



$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} V_{s_1} \\ \vdots \\ \vdots \end{bmatrix}$$

$\underset{\sim}{A}$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = A^{-1} \begin{bmatrix} V_{s_1} \\ \vdots \\ \vdots \end{bmatrix}$$

Figure 7-25: Circuit for Example 7-12 in (a) the time domain and (b) the phasor domain.

Example 7-12: Nodal Analysis

Apply the nodal-analysis method to determine $i_L(t)$ in the circuit of **Fig. 7-25(a)**. The sources are given by:

$$v_{s_1}(t) = 12 \cos 10^3 t \text{ V},$$

$$v_{s_2}(t) = 6 \sin 10^3 t \text{ V}.$$

$V_{s_1} = 12 e^{j0°}$

$V_{s_2} = 6 e^{-j90°}$

Solution: We first demonstrate how to solve this problem using the standard nodal-analysis method (Section 3-2), and then we solve it again by applying the by-inspection method (Section 3-4).

Nodal-analysis method

Our first step is to transform the given circuit to the phasor domain. Accordingly,

$$\mathbf{Z}_C = \frac{1}{j\omega C} = \frac{-j}{10^3 \times 0.25 \times 10^{-3}} = -j4\ \Omega,$$

$$\mathbf{Z}_L = j\omega L = j10^3 \times 10^{-3} = j1\ \Omega,$$

$$v_{s_1} = 12 \cos 10^3 t \quad \leftrightarrow \quad \mathbf{V}_{s_1} = 12\text{ V},$$

and

$$v_{s_2} = 6 \sin 10^3 t \quad \leftrightarrow \quad \mathbf{V}_{s_2} = -j6\text{ V},$$

where for $\mathbf{V}_{s_2}$ we used the property given in **Table 7-2**, namely that the phasor counterpart of sin ωt is $-j$. Using these values, we generate the phasor-domain circuit given in **Fig. 7-25(b)** in

which we selected one of the extraordinary nodes as a ground node and assigned phasor voltages $\mathbf{V}_1$ to $\mathbf{V}_3$ to the other three.

Our plan is to write the voltage node equations at nodes 1 to 3, solve them simultaneously to find $\mathbf{V}_1$ to $\mathbf{V}_3$, and then use the value of $\mathbf{V}_2$ to obtain $\mathbf{I}_L$. The final step will involve transforming $\mathbf{I}_L$ to the time domain to obtain $i_L(t)$.

At node 1, KCL requires that

$$\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 = 0. \qquad (7.103)$$

In terms of node voltages $\mathbf{V}_1$ to $\mathbf{V}_3$,

$$\mathbf{I}_1 = \frac{\mathbf{V}_1 - \mathbf{V}_3}{R_3} = \frac{\mathbf{V}_1 - \mathbf{V}_3}{2},$$

$$\mathbf{I}_2 = \frac{\mathbf{V}_1 - \mathbf{V}_2 + \mathbf{V}_{s_1}}{R_2} = \frac{\mathbf{V}_1 - \mathbf{V}_2 + 12}{2},$$

and

$$\mathbf{I}_3 = \frac{\mathbf{V}_1}{R_1 + \mathbf{Z}_C} = \frac{\mathbf{V}_1}{3 - j4}.$$

Inserting the expressions for $\mathbf{I}_1$ to $\mathbf{I}_3$ in Eq. (7.103) and then rearranging the terms leads to

$$\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{3 - j4}\right)\mathbf{V}_1 - \frac{1}{2}\mathbf{V}_2 - \frac{1}{2}\mathbf{V}_3 = -6. \qquad (7.104)$$

The coefficient of $\mathbf{V}_1$ can be simplified as follows:

$$\begin{aligned} \frac{1}{2} + \frac{1}{2} + \frac{1}{3 - j4} &= 1 + \frac{1}{3 - j4} \\ &= \frac{3 - j4 + 1}{3 - j4} \\ &= \frac{4 - j4}{3 - j4} \times \frac{3 + j4}{3 + j4} \\ &= \frac{(12 + 16) + j(16 - 12)}{9 + 16} = 1.12 + j0.16. \end{aligned} \qquad (7.105)$$

Inserting Eq. (7.105) in Eq. (7.104) and multiplying all terms by 2 leads to the following simplified algebraic equation for node 1:

$$(2.24 + j0.32)\mathbf{V}_1 - \mathbf{V}_2 - \mathbf{V}_3 = -12 \qquad \text{(node 1)}, \quad (7.106)$$

Similarly, at node 2,

$$\frac{\mathbf{V}_2 - \mathbf{V}_1 - 12}{2} + \frac{\mathbf{V}_2}{2 + j1} + \frac{\mathbf{V}_2 - \mathbf{V}_3}{2} = 0,$$

which can be simplified to

$$-\mathbf{V}_1 + (2.8 - j0.4)\mathbf{V}_2 - \mathbf{V}_3 = 12 \qquad \text{(node 2)}, \quad (7.107)$$

and at node 3,

$$\frac{\mathbf{V}_3 - \mathbf{V}_2}{2} + \frac{\mathbf{V}_3 - \mathbf{V}_1}{2} + \frac{\mathbf{V}_3 + j6}{2} = 0,$$

or

$$-\mathbf{V}_1 - \mathbf{V}_2 + 3\mathbf{V}_3 = -j6 \qquad \text{(node 3)}. \quad (7.108)$$

Equations (7.106) to (7.108) now are ready to be cast in matrix form:

$$\begin{bmatrix} (2.24 + j0.32) & -1 & -1 \\ -1 & (2.8 - j0.4) & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \end{bmatrix} = \begin{bmatrix} -12 \\ 12 \\ -j6 \end{bmatrix}. \qquad (7.109)$$

Matrix inversion, either manually or by MATLAB or MathScript software, provides the solution:

$$\mathbf{V}_1 = -(4.72 + j0.88)\text{ V}, \qquad (7.110a)$$

$$\mathbf{V}_2 = (2.46 - j0.89)\text{ V}, \qquad (7.110b)$$

and

$$\mathbf{V}_3 = -(0.76 + j2.59)\text{ V}. \qquad (7.110c)$$

Hence,

$$\mathbf{I}_L = \frac{\mathbf{V}_2}{2 + j1} = \frac{2.46 - j0.89}{2 + j1} = 0.81 - j0.85 = 1.17e^{-j46.5°}\text{ A},$$

and its corresponding time-domain counterpart is

$$\begin{aligned} i_L(t) &= \mathfrak{Re}[\mathbf{I}_L e^{j1000t}] \\ &= \mathfrak{Re}[1.17e^{-j46.4°} e^{j1000t}] = 1.17 \cos(1000t - 46.5°)\text{ A}. \end{aligned} \qquad (7.111)$$

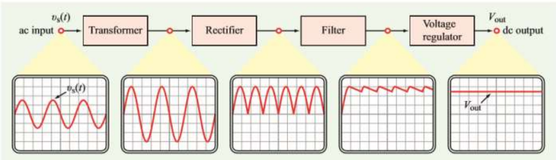


Figure 7-35: Block diagram of a basic dc power supply.

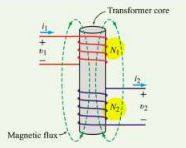
11-2 Transformers

11-2.1 Coupling Coefficient

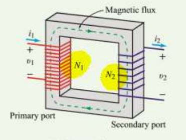
To couple magnetic flux between two coils, the coils may be wound around a common core (**Fig. 11-7(a)**), on two separate arms of a rectangular core (**Fig. 11-7(b)**), or in any other arrangement conducive to having a significant fraction of the magnetic flux generated by each coil shared with the other. The ***coupling coefficient*** k defines the degree of magnetic coupling between the coils, with $0 \leq k \leq 1$. For a ***loosely coupled*** pair of coils, $k < 0.5$; for ***tightly coupled*** coils, $k > 0.5$; and for ***perfectly coupled*** coils, $k = 1$. The magnitude of k depends on the physical geometry of the two-coil configuration and the magnetic permeability μ of the core material.

> ► A transformer is said to be ***linear*** if μ of its core material is a constant, independent of the magnitude of the currents flowing through the coils (and hence, the strength of the induced magnetic field). ◄

Most core materials, including air, wood, and ceramics, are nonferromagnetic, and their μ is approximately equal to μ_0, the ***permeability of free space***. When nonferromagnetic materials are used for the common core around which the coils are wound, the magnitude of k depends entirely on how tightly coupled the two windings are. Such transformers are indeed linear, but the magnitude of k is seldom greater than 0.4. Increasing k requires the use of ferromagnetic cores, but the transformer becomes heavier in weight and its behavior becomes nonlinear. The degree of nonlinearity depends on the choice of materials. *With certain types of purified iron, transformers can be designed to exhibit coupling coefficients approaching unity.*



(a) Cylindrical core



(b) Rectangular core

Figure 11-7: Magnetically coupled coils.

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which can be cast in matrix form as

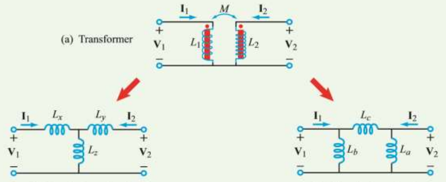
$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}. \quad (11.27c)$$

(transformer)

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} j\omega(L_x + L_z) & j\omega L_z \\ j\omega L_z & j\omega(L_y + L_z) \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}. \quad (11.28)$$

(T-equivalent circuit)

The transformer and its T-equivalent circuit exhibit the same **I**-**V** relationships if the four terms in the matrix of Eq. (11.27)



(b) T-equivalent circuit

(c) Π-equivalent circuit

Figure 11-10: The transformer can be modeled in terms of T- or Π-equivalent circuits.

Transformer dots on same ends

$$L_x = L_1 - M, \quad (11.29a)$$

$$L_y = L_2 - M, \quad (11.29b)$$

and

$$L_z = M. \quad (11.29c)$$

Had the transformer dots been located on opposite ends, the two terms involving M in Eq. (11.27) would have been preceded by minus signs. Consequently, the element values of inductors L_x, L_y, and L_z would be

Transformer dots on opposite ends

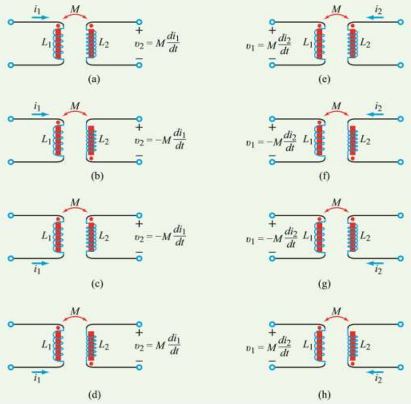
$$L_x = L_1 + M, \quad (11.30a)$$

$$L_y = L_2 + M, \quad (11.30b)$$

and

$$L_z = -M. \quad (11.30c)$$

Even though a negative value for inductance L_z is not physically realizable, the mathematical equivalency holds nonetheless and the equivalent circuit is perfectly applicable.



7-12.1 Ideal Transformers

A transformer consists of two inductors called ***windings***, that are in close proximity to each other but not connected electrically. The two windings are called the ***primary*** and the ***secondary***, as shown in Fig. 7-36. Even though the two windings are isolated electrically—meaning that no current flows between them—when an ac voltage is applied to the primary, it creates a magnetic flux that permeates both windings through a common ***core***, inducing an ac voltage in the secondary.

> ► The ***transformer*** gets its name from the fact that it is used to transform currents, voltages, and impedances between its primary and secondary circuits. ◄

The key parameter that determines the relationships between the primary and the secondary is the ***turns ratio*** $n = N_2/N_1$,

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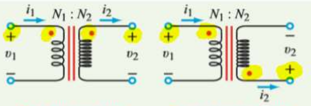
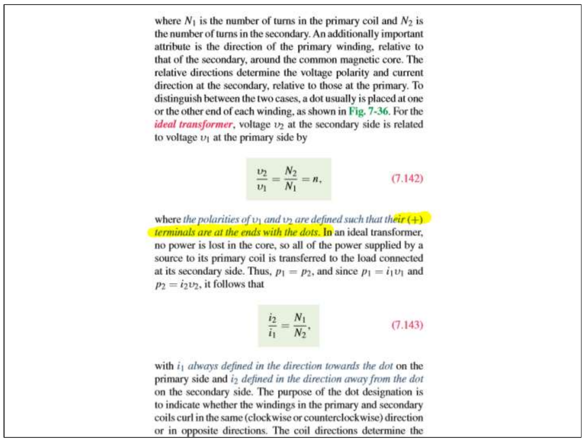


Figure 7-36: Schematic symbol for an ideal transformer. Note the reversal of the voltage polarity and current direction when the dot location at the secondary is moved from the top end of the coil to the bottom end. For both configurations:

$$\frac{v_2}{v_1} = \frac{N_2}{N_1} = n, \quad \frac{i_2}{i_1} = \frac{N_1}{N_2} = \frac{1}{n}, \quad \frac{p_2}{p_1} = \frac{v_2 i_2}{v_1 i_1} = 1$$



where N_1 is the number of turns in the primary coil and N_2 is the number of turns in the secondary. An additionally important attribute is the direction of the primary winding, relative to that of the secondary, around the common magnetic core. The relative directions determine the voltage polarity and current direction at the secondary, relative to those at the primary. To distinguish between the two cases, a dot usually is placed at one or the other end of each winding, as shown in Fig. 7-36. For the ***ideal transformer***, voltage υ_2 at the secondary side is related to voltage υ_1 at the primary side by

$$\frac{\upsilon_2}{\upsilon_1} = \frac{N_2}{N_1} = n, \qquad (7.142)$$

where *the polarities of υ_1 and υ_2 are defined such that their (+) terminals are at the ends with the dots.* In an ideal transformer, no power is lost in the core, so all of the power supplied by a source to its primary coil is transferred to the load connected at its secondary side. Thus, $p_1 = p_2$, and since $p_1 = i_1\upsilon_1$ and $p_2 = i_2\upsilon_2$, it follows that

$$\frac{i_2}{i_1} = \frac{N_1}{N_2}, \qquad (7.143)$$

with i_1 *always defined in the direction towards the dot* on the primary side and i_2 *defined in the direction away from the dot* on the secondary side. The purpose of the dot designation is to indicate whether the windings in the primary and secondary coils curl in the same (clockwise or counterclockwise) direction or in opposite directions. The coil directions determine the

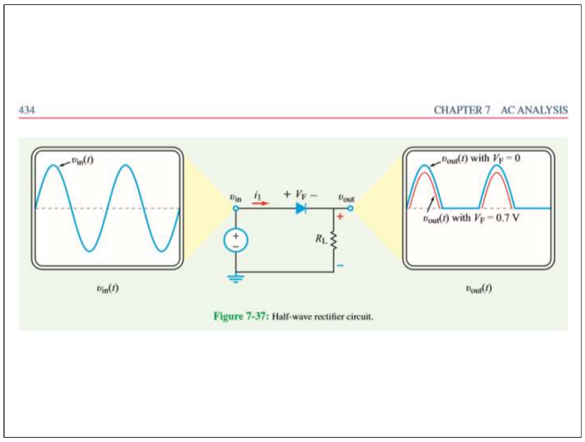
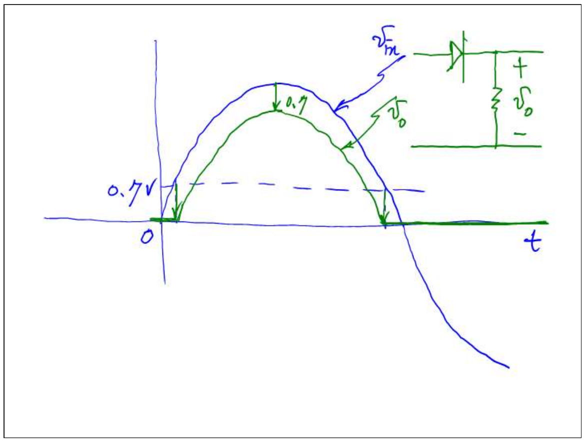
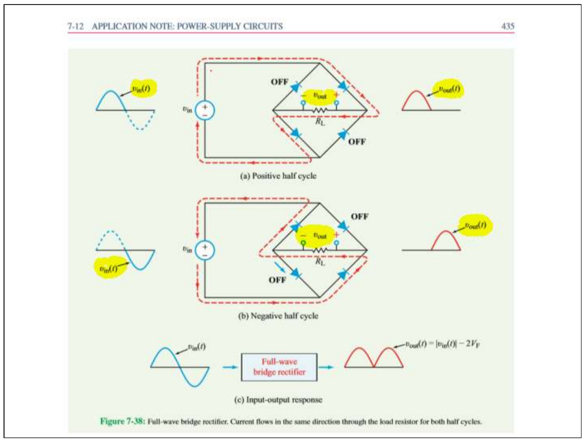


Figure 7-37: Half-wave rectifier circuit.



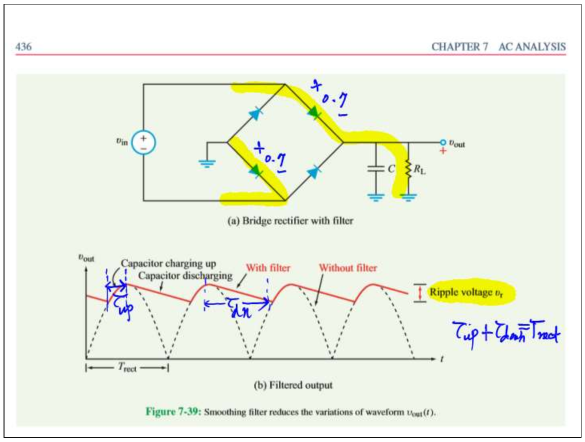


(a) Positive half cycle

(b) Negative half cycle

(c) Input-output response

Figure 7-38: Full-wave bridge rectifier. Current flows in the same direction through the load resistor for both half cycles.

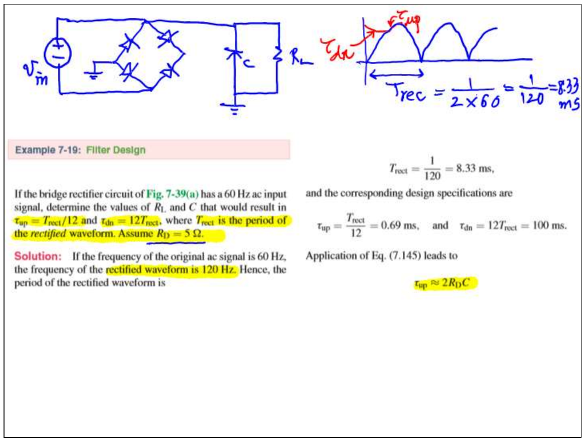


(a) Bridge rectifier with filter

$\tau_{up} + \tau_{dn} = T_{rect}$

(b) Filtered output

Figure 7-39: Smoothing filter reduces the variations of waveform $\upsilon_{out}(t)$.



$T_{rec} = \frac{1}{2 \times 60} = \frac{1}{120} = 8.33$ ms

Example 7-19: Filter Design

If the bridge rectifier circuit of Fig. 7-39(a) has a 60 Hz ac input signal, determine the values of R_L and C that would result in $\tau_{up} = T_{rect}/12$ and $\tau_{dn} = 12T_{rect}$, where T_{rect} is the period of the *rectified* waveform. Assume $R_D = 5\ \Omega$.

Solution: If the frequency of the original ac signal is 60 Hz, the frequency of the rectified waveform is 120 Hz. Hence, the period of the rectified waveform is

$$T_{rect} = \frac{1}{120} = 8.33 \text{ ms},$$

and the corresponding design specifications are

$$\tau_{up} = \frac{T_{rect}}{12} = 0.69 \text{ ms}, \quad \text{and} \quad \tau_{dn} = 12T_{rect} = 100 \text{ ms}.$$

Application of Eq. (7.145) leads to

$$\tau_{up} \approx 2R_D C$$

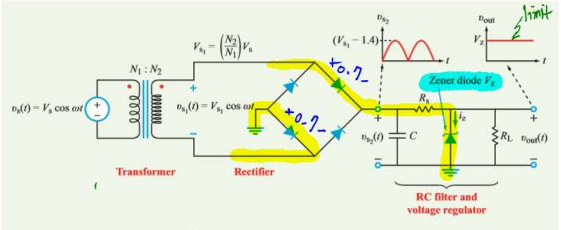


Figure 7-40: Complete power-supply circuit.

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or

$$C = \frac{\tau_{up}}{2R_D} = \frac{0.69 \times 10^{-3}}{2 \times 5} = 69\ \mu\text{F}.$$

With the value of C known, application of Eq. (7.146) gives

$$R_L = \frac{\tau_{dn}}{C} = \frac{100 \times 10^{-3}}{69 \times 10^{-6}} = 1.45\ \text{k}\Omega.$$

7-12.4 Voltage Regulator

The circuit shown in Fig. 7-40 includes all of the power-supply subcircuits we have discussed thus far, plus two additional elements, namely a series resistance R_s and a *zener diode*. When operated in reverse breakdown, the zener diode maintains the voltage across it at a constant level V_z—so long as the current i_z passing through it remains between certain limits. Since the diode is connected in parallel with R_L, the output voltage becomes equal to the *zener voltage* V_z, and the effective time constant of the smoothing filter becomes $\tau = R_s C$. It is worth noting that the addition of the zener diode reduces the *peak-to-peak ripple voltage* V_r (Fig. 7-39(b)) at the output of the RC filter by about an order of magnitude. An approximate expression for the peak-to-peak ripple voltage with the zener diode in place is given by

$$V_r = \frac{[(V_{s_1} - 1.4) - V_z]T_{rect}}{R_s C} \times \frac{(R_z \parallel R_L)}{R_s + (R_z \parallel R_L)}, \quad (7.147)$$

where V_{s_1} is the amplitude of the ac signal at the output of the transformer (Fig. 7-40), the factor 1.4 V accounts for the voltage drop across a pair of diodes in the rectifier, V_z is the manufacturer-rated zener voltage for the specific model used in the circuit, T_{rect} is the period of the rectified waveform, and R_z is the manufacturer specified value of the *zener-diode resistance*.

Example 7-20: Power-Supply Design

A power supply with the circuit configuration shown in Fig. 7-40 has the following specifications: the input voltage is 60 Hz with an rms amplitude $V_{rms} = 110$ V where $V_{rms} = V_s/\sqrt{2}$ (the rms value of a sinusoidal function is

Time constant of the smoothing filter

$$\tau = R_s C \overset{\uparrow}{=} 50 \times 69 \times 10^{-6} = 3450 \times 10^{-6} = 3.45\ \text{ms}$$

If $R_s = 50\Omega$

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discussed in Chapter 8), $N_1/N_2 = 5$, $C = 2$ mF, $R_s = 50\ \Omega$, $R_L = 1\ \text{k}\Omega$, $V_z = 24$ V, and $R_z = 20\ \Omega$. Determine v_{out}, the ripple voltage, and the ripple fraction relative to v_{out}.

Solution: At the secondary side of the transformer,

$$v_{s_1}(t) = \left(\frac{N_2}{N_1}\right)(V_s \cos 377t)$$

$$= \frac{1}{5} \times 110\sqrt{2} \cos 377t = 31.11 \cos 377t\ \text{V}.$$

Hence, $V_{s_1} = 31.11$ V, which is greater than the zener voltage $V_z = 24$ V.

Consequently, the zener diode will limit the output voltage at

$$v_{out} = V_z = 24\ \text{V}.$$

In Example 7-19, we established that $T_{rect} = 8.33$ ms. Also,

$$R_z \parallel R_L = \frac{20 \times 1000}{20 + 1000} = 19.6\ \Omega.$$

Application of Eq. (7.147) gives

$$V_r = \frac{[(V_{s_1} - 1.4) - V_z]T_{rect}}{R_s C} \times \left[\frac{(R_z \parallel R_L)}{R_s + (R_z \parallel R_L)}\right]$$

$$= \frac{[(31.11 - 1.4) - 24]}{50 \times 2 \times 10^{-3}}(8.33 \times 10^{-3}) \times \frac{19.6}{50 + 19.6}$$

$$= 0.13\ \text{V (peak-to-peak)}.$$

Hence,

$$\text{ripple fraction} = \frac{(V_r/2)}{V_z} = \frac{0.13/2}{24} = 0.0027,$$

which represents a relative variation of less than ±0.3 percent.