Sinusoidal waveforms can be expressed in terms of either sine or cosine functions.

To avoid confusion, we adopt the cosine form as our reference standard throughout this and followup chapters.

This means that we will always express voltages and currents in terms of cosine functions, so if a voltage (or current) waveform is given in terms of a sine function, we should first convert it to a cosine form with a positive amplitude before proceeding with our circuit analysis. Conversion from sine to cosine form is realized through the application of Eq. (7.7a) of Table 7-1.

Example 7-2: Phase Lead/Lag

Given the current waveforms

\[ i_1(t) = 3 \cos (\omega t - 30^\circ) \]  
and

\[ i_2(t) = 3 \cos (\omega t + 45^\circ) \]

does \( i_1(t) \) lead \( i_2(t) \), or the other way around, and by how much?

Solution: Standard cosine form requires that the sinusoidal functions be cosines and that the amplitudes have positive values. Application of Eq. 7.7a of Table 7-1 allows us to remove the negative sign preceding the amplitude of \( i_1(t) \),

\[ 3 \cos (\omega t - 30^\circ) = 3 \cos (\omega t + 120^\circ) \]

Application of Eq. (7.7a) to \( i_2(t) \) leads to

\[ 3 \cos (\omega t + 45^\circ) = 3 \cos (\omega t + 150^\circ) \]

Hence, \( \phi_1 = 150^\circ \) and \( \phi_2 = -45^\circ \), and

Exercise 7-2: Given two current waveforms:

\[ i_1(t) = 3 \cos \omega t \]  
and

\[ i_2(t) = 3 \sin (\omega t + 36^\circ) \]

does \( i_2(t) \) lead or lag \( i_1(t) \), and by what phase angle?

Answer: \( i_2(t) \) lags \( i_1(t) \) by 54°. (See CABS.)

\[ \phi_2(\omega t + 36^\circ) = 3 \cos (\omega t + 54^\circ) \]
A complex number $z$ may be written in the **rectangular form**

$$ z = x + jy, \quad (7.12) $$

where $x$ and $y$ are the *real* (Re) and *imaginary* (Im) parts of $z$, respectively, and $j = \sqrt{-1}$. That is,

$$ x = \text{Re}(z), \quad y = \text{Im}(z). \quad (7.13) $$

Alternatively, $z$ may be written in **polar form** as

$$ z = r e^{j\theta} = |z| e^{j\theta}, \quad (7.14) $$

where $|z|$ is the magnitude of $z$, $\theta$ is its phase angle, and the form $e^{j\theta}$ is a useful shorthand representation commonly used in numerical calculations. A phase angle may be expressed in degrees, as in $\theta = 30^\circ$, or in radians, as in $\theta = 0.52$ rad.

By applying *Euler's identity*,

$$ e^{j\theta} = \cos \theta + j \sin \theta, \quad (7.15) $$

we can convert $z$ from polar form, as in Eq. (7.14), into rectangular form, as in Eq. (7.12),

$$ z = |z| e^{j\theta} = |z| \cos \theta + j|z| \sin \theta, \quad (7.16) $$

### Table 7.1: Properties of complex numbers.

- **Euler’s Identity:** $e^{j\theta} = \cos \theta + j \sin \theta$
- **Complex Conjugate:** $z = z^*$
- **Real Part:** $\text{Re}(z)$
- **Imaginary Part:** $\text{Im}(z)$
- **Phase Angle:** $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

#### 3.8.1 Time Domain / Frequency Domain Correspondence

Transformation from the time domain to the frequency domain involves transforming different quantities in the complex Fourier transform. This transformation must be effective to achieve proper scaling. This scaling is done through the use of exponential frequency spectrum. The values of the voltages and currents under this transformation should be treated as a transformation that involves differentiation or integration with respect to $t$. Any such quantities involving the use of time in the complex Fourier transform must be treated as a function of $e^{j\omega t}$, where $\omega$ is the angular frequency of the function.

### Formulae

**Time Domain**

$$ v(t) = V \cos(\omega t + \phi), \quad (7.18) $$

**Frequency Domain**

$$ V = V \cos(\omega t + \phi), \quad (7.19) $$

for $t = 0$, which may be generalized.

**Exponentials**

$$ e^{j\omega t} = V \cos(\omega t + \phi), \quad (7.20) $$

Occasionally, when one encounters a function transforming into or out of the frequency domain, the function under the input of time must be a complex function of frequency. This function may be written as $e^{j\omega t}$, where $\omega$ is the angular frequency of the function.

**Example**: Convert the following time-domain function $v(t) = V \cos(\omega t + \phi)$ to the frequency domain.

$$ v(t) = V \cos(\omega t + \phi), \quad (7.21) $$

Finally, for complex numbers, by definition, not a function of time, the dynamics of $\frac{d}{dt}$ is governed.
\[ X = X e^{j\theta} \]
\[ X e^{j\omega t} = X e^{j(\omega t + \phi)} = X \cos(\omega t + \phi) + jX \sin(\omega t + \phi) \]
\[ \text{Re} \ X e^{j\omega t} = X \cos(\omega t + \phi) \]
\[ \text{Im} \ X e^{j\omega t} = X \sin(\omega t + \phi) \]

Table 7.6: Time-domain counterparts of the following waveforms:

<table>
<thead>
<tr>
<th>Property</th>
<th>( X )</th>
<th>( \omega t )</th>
<th>( \phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V )</td>
<td>( V )</td>
<td>( 2\pi f t )</td>
<td>( \phi )</td>
</tr>
<tr>
<td>( I )</td>
<td>( I )</td>
<td>( 2\pi f t )</td>
<td>( \phi )</td>
</tr>
<tr>
<td>( \omega )</td>
<td>( \omega )</td>
<td>( 2\pi f t )</td>
<td>( \phi )</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>( \Delta )</td>
<td>( 2\pi f t )</td>
<td>( \phi )</td>
</tr>
<tr>
<td>Frequency</td>
<td>( f )</td>
<td>( f )</td>
<td>( f )</td>
</tr>
</tbody>
</table>

Exercise 7.6: Determine the phase counterparts of the following waveforms:

(a) \( v(t) = 2 \sin(\omega t - 30^\circ) \) A
(b) \( v(t) = 2 \cos(\omega t + 30^\circ) \) A

Exercise 7.7: Obtain the time-domain waveforms (in standard cosine format) corresponding to the following phasors at angular frequency \( \omega = 3 \times 10^{3} \) rad/s:

(a) \( V_1 = 10 \sin(\omega t - 30^\circ) \) V
(b) \( V_2 = 10 \cos(\omega t + 30^\circ) \) V

Exercise 7.8: At \( \omega = 10^{3} \) rad/s, the phasor voltage across and current through a certain element are given by \( V = 120 \text{ V} \) and \( I = 200 \text{ A} \). What type of element is it?

Answer: Capacitor with \( C = 0.5 \mu \text{F} \). (See 25)
Using the specified values, namely \( R = \sqrt{3} \Omega \), \( C = 1 \mu F \), and \( \omega = 10^3 \) rad/s, Eq. (7.10) becomes

\[
I = i_R = j \frac{10^3}{\sqrt{3} + j \omega C} \text{ ampere}
\]

\[
I = i_L = j \frac{10^3}{\sqrt{3} + j \omega L} \text{ ampere}
\]

\[
I = i_C = j \frac{10^3}{\sqrt{3} + j \omega R} \text{ ampere}
\]

To obtain the independent component form (as in polar form), where \( Z \) is in polar form and its magnitude is specified, a similar technique is used to divide two complex numbers using the polar form. To that end, we should replace \( j \) in the numerator with \( e^{j\theta} \) and convert the denominator into polar form.

\[
\begin{align*}
\text{Current Division} & \quad \text{Voltage Division} \\
\text{Exercise 7.9: Determine the input impedance at } \omega = 10^3 \text{ rad/s for each of the circuits in Fig. 7.7.9.} & \quad \text{Figure 6.10: Voltage division among two impedances in series.}
\end{align*}
\]
**Solution:** (a) We start by converting $v_0(t)$ to cosine format:

$$v_0(t) = 4 \sin(10^5 t + 15^\circ)$$

$$= 4 \cos(10^5 t + 15^\circ - 90^\circ) = 4 \cos(10^5 t - 75^\circ) \text{ V}.$$ 

The corresponding phasor voltage is

$$V_\Phi = 4e^{-j75^\circ} \text{ V},$$

and the impedances shown in Fig. 7-13(b) are given by

$$Z_{R_1} = R_1 = 10 \Omega,$$

$$Z_C = \frac{-j}{\alpha C} = \frac{-j}{10^5 \times 10^{-8}} = -j10 \Omega,$$
\( \Delta \rightarrow Y \) transformation:

\[
Z_1 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c},
\]

\( \Delta \rightarrow Y \) transformation:

\[
Z_2 = \frac{Z_a Z_c}{Z_a + Z_b + Z_c},
\]

\[
Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}.
\]

\[
Z_1 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c} = \frac{-j6 \times 12}{24 - j12 - j6 + 12} = \frac{-j72}{36 - j18} = (0.8 - j1.6) \Omega,
\]

\[
Z_2 = \frac{Z_a Z_c}{Z_a + Z_b + Z_c} = \frac{(24 - j12) \times 12}{36 - j18} = 8 \Omega,
\]

and

\[
Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c} = \frac{-j6(24 - j12)}{36 - j18} = -j4 \Omega.
\]

In Fig. 7-15(c), \( Z_d \) represents the series combination of \( Z_3 \) and \( Z_3 \).
Open-circuit / short-circuit method

\[ Z_{Th} = \frac{V_{oc}}{I_{oc}}, \]  

(7.83)

where \( I_{oc} \) is the short-circuit current at the circuit's output terminals (Fig. 7.18(a)).

Exercise 7.15: Write down the node-voltage matrix equation for the circuit in Fig. 7.15.

[Diagram of circuit with labels and equation: \( \begin{bmatrix} (2 - j2) & -2 + j2 & V_1 \\ -2 + j2 & (2 - j2) & V_2 \\ \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix} \).]

Answer: \( \begin{bmatrix} (2 - j2) & -2 + j2 & V_1 \\ -2 + j2 & (2 - j2) & V_2 \\ \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix} \).

Example 7.12: Nodal Analysis

Apply the nodal-analysis method to determine \( i_c(t) \) in the circuit of Fig. 7.20(a). The sources are given by:

\( v_{in}(t) = 12 \cos 10^5 t \) V,
\( v_{in}(t) = 6 \sin 10^5 t \) V.

Solution: We first demonstrate how to solve this problem using the standard nodal-analysis method (Section 3.2), and then we solve it again by applying the by-inspection method (Section 3.6).
Nodal-analysis method

Our first step is to transform the given circuit to the phasor domain. Accordingly,

\[ Z_C = \frac{1}{j\omega C} = \frac{-j}{10^3 \times 0.25 \times 10^{-3}} = -j4 \ \Omega, \]

\[ Z_L = j\omega L = j10^3 \times 10^{-3} = j1 \ \Omega, \]

\[ v_{v1} = 12 \cos 10^3 t \quad \Rightarrow \quad V_{v1} = 12 \ V, \]

and

\[ v_{v2} = 6 \sin 10^3 t \quad \Rightarrow \quad V_{v2} = -j6 \ V, \]

where for \( V_{v2} \) we used the property given in Table 7.2, namely that the phasor counterpart of \( \sin \) of \( \omega t \) is \( -j \). Using these values, we generate the phasor-domain circuit given in Fig. 7-25(b) in...
7.12.1 Ideal Transformers

A transformer consists of two inductors called windings, that are in close proximity to each other but not connected electrically. The two windings are called the primary and the secondary, as shown in Fig. 7.36. Even though the two windings are isolated electrically—meaning that no current flows between them—when an ac voltage is applied to the primary, it creates a magnetic flux that permeates both windings through a common core, inducing an ac voltage in the secondary.

- The transformer gets its name from the fact that it is used to transform currents, voltages, and impedances between its primary and secondary circuits.

The key parameter that determines the relationships between the primary and the secondary is the turns ratio \( n = N_2 / N_1 \).

where \( N_1 \) is the number of turns in the primary coil and \( N_2 \) is the number of turns in the secondary. An additional important attribute is the direction of the primary winding, relative to that of the secondary, around the common magnetic core. One important distinction is that when the primary current is positive, the flux produced is in the same direction for both windings. However, for the ideal transformer, voltage \( v_s \) at the secondary is related to voltage \( v_p \) at the primary side by:

\[
\frac{v_p}{v_s} = n
\]

where the polarities of \( v_p \) and \( v_s \) are defined such that voltage \( v_p \) is positive when viewed from the transformer's secondary coil. If the current in the primary coil is positive, all of the power supplied by the source to the primary coil is transferred to the load connected at the secondary coil. Thus, \( i_p = i_s \), and since \( i_s = i_p/n \), and \( i_p = i_1 \), it follows that:

\[
\frac{i_1}{i_2} = n
\]

where \( i_1 \) is always defined in the direction towards the dot on the primary side and \( i_2 \) is defined in the direction away from the dot on the secondary side. The purposes of the dot designation is to indicate whether the windings in the primary and secondary coils have the same (clockwise or counterclockwise) direction or in opposite directions. The coil directions determine the

---

Figure 7.36: Schematic symbol for an ideal transformer. Notice the reversal of the voltage polarity and current direction when the dot location at the secondary is moved from the top end of the coil to the bottom end. For both configurations:

\[
\frac{v_1}{v_2} = \frac{I_1}{I_2} \quad \frac{v_1}{I_1} = \frac{v_2}{I_2} = n
\]

Figure 7.37: half-wave rectifier circuit.