RLC Circuits

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Series RLC

Parallel RLC

(a) Circuit

(b) At \( t = 0 \), \( C \) acts like an open circuit and \( L \) like a short circuit

Serial RLC Circuit

In order to find \( \dot{x}_L(0) \) and \( \dot{v}_C(0) \), consider:

\[
\begin{align*}
\dot{v}_C(t) &= v_C(t) - R\dot{x}_L(t) - \frac{1}{C}\frac{dC}{dt}
\end{align*}
\]

\[
\begin{align*}
\dot{v}_C(t) &= v_C(t) - R\dot{x}_L(t)
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\]

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\dot{v}_C(t) = \frac{v_C(t) - R\dot{x}_L(t)}{C}
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\]
\[
\text{Laplace transform of \(v_c(t)\) is } V_C(s) = \frac{V_{in}}{s} - \frac{V_{out}}{s}.
\]

**Table 12-6: Properties of the Laplace transform**

<table>
<thead>
<tr>
<th>Property</th>
<th>(f(t))</th>
<th>(F(s))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Linearity</td>
<td>(a_1 f_1(t) + a_2 f_2(t))</td>
<td>(a_1 F_1(s) + a_2 F_2(s))</td>
</tr>
<tr>
<td>2. Time-shifting</td>
<td>(f(t - T))</td>
<td>(e^{-sT}F(s))</td>
</tr>
<tr>
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<td>(af(t))</td>
<td>(F(a^{-1}s))</td>
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<tr>
<td>4. Time-derivative</td>
<td>(f'(t))</td>
<td>(sF(s) - f(0))</td>
</tr>
<tr>
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<td>(\int_0^t f(\tau) d\tau)</td>
<td>(\frac{1}{s}F(s))</td>
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**Properties of the differentiator**

\[
\frac{d}{dt} x(t) = \frac{d}{ds} X(s) \Big|_{s=0}
\]

\[
L\left\{\frac{d}{dt} x(t)\right\} = sX(s) - x(0) = \frac{d}{ds} X(s)
\]

**Properties of the integrator**

\[
\int_0^t x(\tau) d\tau = \frac{1}{s} X(s) \Big|_{s=0}
\]

\[
L\left\{\int_0^t x(\tau) d\tau\right\} = \frac{1}{s} X(s)
\]

\[
\frac{d}{ds} \left(\frac{1}{s} X(s)\right) = \frac{d}{dt} x(t)
\]

\[
\left(\frac{d}{dt} x(t)\right) = \frac{d}{ds} X(s) \Big|_{s=0}
\]

\[
\frac{d}{ds} X(s) = \frac{d}{dt} x(t)
\]

\[
X(s) = \frac{V_L(s)}{sL} + \frac{V_C(s)}{sC}
\]

\[
\Rightarrow \quad i_L(s) = -\frac{V_L(s)}{L} + \frac{V_C(s)}{C}
\]

\[
\frac{V_L(s)}{L} + \frac{V_C(s)}{C} = 0
\]

\[
\text{For } s > 0, L \text{ is inductive, } C \text{ is capacitive.}
\]

\[
\Delta^2 + \frac{\Delta}{L} + \frac{1}{LC} = 0
\]

\[
\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
\Delta = \frac{-\frac{1}{L} \pm \sqrt{\left(-\frac{1}{L}\right)^2 - 4 \cdot \frac{1}{LC}}}{2}
\]

\[
\text{where, } a = 1, \quad b = -\frac{1}{L}, \quad c = \frac{1}{LC}
\]

\[
\frac{\Delta}{L} = \frac{-\frac{1}{L} \pm \sqrt{\left(-\frac{1}{L}\right)^2 - 4 \cdot \frac{1}{LC}}}{2 \cdot \frac{1}{L}}
\]

\[
= \left(\frac{1}{L} \pm \frac{1}{\sqrt{LC}}\right)
\]

\[
\text{If } \Delta = \frac{-\frac{1}{L} + \frac{1}{\sqrt{LC}}}{2} = \text{critically damped}
\]

\[
\text{If } \Delta = \frac{-\frac{1}{L} - \frac{1}{\sqrt{LC}}}{2} = \text{underdamped}
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\]
6.3 Series RLC Overdamped Response ($\omega > \omega_0$)

A key outcome from the analysis described above is that after closing the switch in a series RLC circuit, the voltage across the capacitor will change or discharge down to equilibrium in the voltage across the source. In this section, we derive the differential equation for the series RLC circuit in Fig. 6.7 and then solve it to obtain an expression for $V(t)$ for $\omega > \omega_0$ with $t > 0$ designated as the time immediately after the switch is closed.

As noted in the preceding section, the nature of the solution for $\omega > \omega_0$ depends on how the magnitude of the damping coefficient $\zeta$ compares with that of the resonant frequency $\omega_0$. The values of the two parameters are dictated by the values of $R$, $L$, and $C$, per the expressions in Eq. (6.1). In the present section, we consider the case corresponding to $\omega > \omega_0$, which is called the overdamped response. The other two cases are treated in follow-up sections.

6.3.1 Differential Equation

For the circuit in Fig. 6.7, the KVL loop equation for $t \geq 0$ (after closing the switch) is

$$Ri_C + L \frac{di_C}{dt} + V_C = V_s \quad \text{for} \ t \geq 0, \quad (6.2)$$

where $i_C$ and $V_C$ are the current through and voltage across the capacitor. The capacitor may or may not have had charge on it. If it had, we denote the value of the initial voltage across it $V_C(0)$, which is the same as $V_C(0^-)$, the voltage across it before closing the switch (since the voltage across a capacitor cannot change instantaneously).
6-3.2 Solution of Differential Equation

The general solution of the second-order differential equation given by Eq. (6.5) consists of two components:

\[ u_C(t) = u_{th}(t) + u_{ps}(t), \]  
(6.7)

where \( u_{th}(t) \) is the transient (also called homogeneous) solution of Eq. (6.5) or the natural response of the RLC circuit and \( u_{ps}(t) \) is the steady-state solution (also called particular solution). The transient solution is the solution of Eq. (6.5) under source-free conditions, i.e., with \( V_s = 0 \), which means that \( c = V_s/LC \) also is zero. Thus \( u_{th}(t) \) is the solution of

\[ a u'' + b u' + c u = 0 \quad \text{(source-free)}, \]  
(6.8)

The steady-state solution \( u_{ps}(t) \) is related to the forcing function on the right-hand side of Eq. (6.5), and its functional form is similar to that of the forcing function. Since in the present case, the forcing function \( c \) is simply a constant, so is \( u_{ps}(t) \). That is, \( u_{ps}(t) \) is a non-time-varying constant \( u_p \) that will be determined later from initial and final conditions. Moreover, as we will see shortly, the transient component \( u_{th}(t) \) always goes to zero as \( t \to \infty \) (that's why it is called transient). Hence, as \( t \to \infty \), Eq. (6.7) reduces to

\[ u_C(\infty) = u_{ps}, \]  
(6.9)

in which case Eq. (6.7) can be rewritten as

\[ u_C(t) = u_{th}(t) + u_C(\infty). \]  
(6.10)

Our remaining task is to determine \( u_{th}(t) \).

equations. Thus, we assume that

\[ u_{th}(t) = Ae^{at}, \]  
(6.11)

where \( A \) and \( a \) are constants to be determined later. To ascertain that Eq. (6.11) is indeed a viable solution of Eq. (6.8), we insert the proposed expression for \( u_{th}(t) \) and its first and second derivatives in Eq. (6.8). The result is

\[ s^2 A e^{at} + a s A e^{at} + b A e^{at} = 0, \]  
which simplifies to

\[ s^2 + as + b = 0. \]  
(6.13)

Hence, the proposed solution given by Eq. (6.11) is indeed an acceptable solution so long as Eq. (6.13) is satisfied.

The quadratic equation given by Eq. (6.13) is known as the characteristic equation of the differential equation. It has two roots:

\[ s_1 = \frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b}, \]  
(6.14a)

\[ s_2 = \frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - b}. \]  
(6.14b)

The solution in the present section pertains to the overdamped case corresponding to \( a > \omega_0 \). Under this condition, both \( s_1 \) and \( s_2 \) are real, negative numbers. Consequently, as \( t \to \infty \), the first two terms in Eq. (6.16) go to zero, just as we asserted earlier.

\[ u_{th}(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{for} \quad t \geq 0, \]  
(6.15)

where constants \( A_1 \) and \( A_2 \) are to be determined shortly. Inserting Eq. (6.15) into Eq. (6.10) leads to

\[ u_C(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + u_C(\infty). \]  
(6.16)

The exponential coefficients \( s_1 \) and \( s_2 \) are given by Eq. (6.14) in terms of constants \( a \) and \( b \), both of which are defined in Eq. (6.6). By reintroducing the damping coefficient \( \alpha \) and resonant frequency \( \omega_0 \), which we defined earlier in Eq. (6.1),

\[ \alpha = \frac{R}{2L}, \]  
(6.17a)

\[ \omega_0 = \frac{1}{\sqrt{LC}}. \]  
(6.17b)

the expressions given by Eq. (6.14) become

\[ s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \]  
(6.18a)

\[ s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}. \]  
(6.18b)

\[ A_1 = \frac{\frac{1}{c} \int_{0}^{\infty} e^{-\alpha t} dt - s_2 [u_C(0) - u_C(\infty)]}{s_1 - s_2}, \]  
(6.22a)

\[ A_2 = \frac{\frac{1}{c} \int_{0}^{\infty} e^{-\alpha t} dt - s_1 [u_C(0) - u_C(\infty)]}{s_2 - s_1}. \]  
(6.22b)

This concludes the general solution for the overdamped response. A summary of relevant expressions is available in Table 6-1.
### Table 6-1: Step response of RLC circuit

<table>
<thead>
<tr>
<th>Series RLC</th>
<th>Parallel RLC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_L = \frac{L}{R}$</td>
<td>$X_L = \frac{1}{RC}$</td>
</tr>
<tr>
<td>$V_R = V_0 e^{-\frac{t}{R}}$</td>
<td>$V_R = V_0 e^{-\frac{t}{RC}}$</td>
</tr>
<tr>
<td>$I_L = \frac{V_0}{R} e^{-\frac{t}{R}}$</td>
<td>$I_L = \frac{V_0}{1 + \frac{1}{RC}} e^{-\frac{t}{RC}}$</td>
</tr>
<tr>
<td>$V_C = \frac{V_0}{1 + \frac{1}{RC}} e^{-\frac{t}{RC}}$</td>
<td>$V_C = \frac{V_0}{R} e^{-\frac{t}{R}}$</td>
</tr>
</tbody>
</table>

### Clocked Parallel RLC

- $\phi(t) = V_0 e^{-\frac{t}{R}}$
- $\gamma(t) = V_0 e^{-\frac{t}{RC}}$
- $\delta(t) = 0$
- $\eta(t) = 0$

### Overdamped Case

**Given** that in the circuit of Fig. 6-8(a), $V_0 = 16 \ V$, $R = 64 \ \Omega$, $L = 0.8 \ \text{H}$, and $C = 2 \ \text{mF}$, determine $V_C(t)$ and $I_C(t)$ for $t \geq 0$.

The capacitor had no charge prior to $t = 0$.

\[
V_C(\infty) = \frac{V_0}{1 + \frac{1}{RC}} = 16 \ V
\]

\[
\alpha = \frac{R}{2L} = \frac{64}{2 \times 0.8} = 40 \ \text{Np/s},
\]

\[
\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.8 \times 2 \times 10^{-3}}} = 25 \ \text{rad/s}.
\]

**\(\omega > \omega_0\) (overdamped case)**

Prior to $t = 0$, there was no current in the circuit, and since the current through $L$ (which is also the current through $C$) cannot change instantaneously, it follows that

\[I_L(0) = I_C(0) = I_C(0^-) = 0.\]

From Eq. (6.22), $A_1$ and $A_2$ are given by

\[
A_1 = \frac{1}{s + \frac{R}{2L}} \left[ I_L(0) - s I_C(0) - s I_C(0^-) \right] = s I_C(0^-) = -8.8 + 71.2 = 18.25 \ \text{V},
\]

\[
A_2 = \frac{1}{s + \frac{R}{2L}} \left[ I_L(0) - s I_C(0) - s I_C(0^-) \right] = s I_C(0^-) = -8.8 + 71.2 = 22.5 \ \text{V}.
\]

The total response $V_C(t)$ is then given by

\[
V_C(t) = \left[ -18.25 e^{-8.8t} + 22.5 e^{-71.2t} + 16 \right] \ \text{V}
\]

(for $t \geq 0$)