EE 101 Lecture 14, Feb 21, 2019

Quiz 7 on Feb 26 based on HW 7.

[1] Prob. 6.1

[2] 6.2

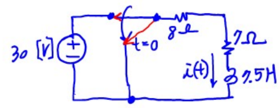
[3] 6.3

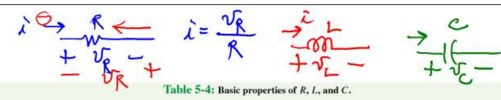
[4] 6.4

[5] 6.5

[6] 6.7

[7] Find $i(t)$, $t>0$ for the circuit below





$i = \frac{v_R}{R}$

Table 5-4: Basic properties of R, L, and C.

Property	R	L	C
i–v relation	$i = \frac{v}{R}$	$i = \frac{1}{L}\int_{t_0}^{t} v\,dt' + i(t_0)$	$i = C\frac{dv}{dt}$
v-i relation	$v = iR$	$v = L\frac{di}{dt}$	$v = \frac{1}{C}\int_{t_0}^{t} i\,dt' + v(t_0)$
p (power transfer in)	$p = i^2R$	$p = Li\frac{di}{dt}$	$p = Cv\frac{dv}{dt}$
w (stored energy)	0	$w = \frac{1}{2}Li^2$	$w = \frac{1}{2}Cv^2$
Series combination	$R_{eq} = R_1 + R_2$	$L_{eq} = L_1 + L_2$	$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$
Parallel combination	$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$	$\frac{1}{L_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$	$C_{eq} = C_1 + C_2$
dc behavior	no change	short circuit	open circuit
Can v change instantaneously?	yes	yes	no
Can i change instantaneously?	yes	no	yes

DC source

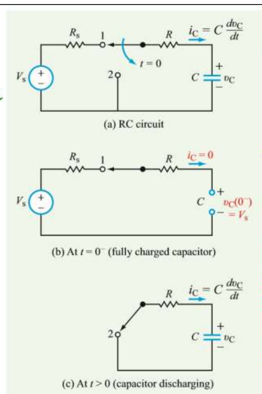


Figure 5-28: RC circuit with an initially charged capacitor that starts to discharge its energy after $t = 0$.

for $t<0$ switch at position 1

In dc, $C\frac{dv_C}{dt} = 0$ open circuit

⇒ no voltage drop across R

thus $v_C(0^-) = V_S$

$t>0$

$$\boxed{C\frac{dv_C}{dt} = -\frac{v_C}{R} \qquad v_C(0^-) = V_S}$$

$$RC\,\frac{dv_C}{dt} + v_C = 0. \qquad (5.69)$$

Upon dividing both terms by RC, Eq. (5.69) takes the form

$$\boxed{\frac{dv_C}{dt} + a v_C = 0 \qquad \text{(source-free)},} \qquad (5.70)$$

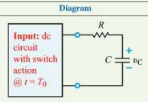
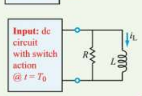
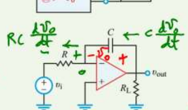
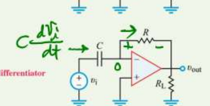
$v_C(0^-) = V_S$ (i.c.)

where

$$\boxed{\frac{dx}{dt} + ax = 0}$$

$$a = \frac{1}{RC}. \qquad (5.71)$$

Table 5-5: Response forms of basic first-order circuits.

Circuit	Diagram	Response
RC		$v_C(t) = \left[v_C(\infty) + [v_C(T_0) - v_C(\infty)]e^{-(t-T_0)/\tau}\right] u(t - T_0)$ $(\tau = RC)$
RL		$i_L(t) = \left[i_L(\infty) + [i_L(T_0) - i_L(\infty)]e^{-(t-T_0)/\tau}\right] u(t - T_0)$ $(\tau = L/R)$
Ideal integrator		$v_{out}(t) = -\frac{1}{RC}\int_{t_0}^{t} v_i\,dt' + v_{out}(t_0)$
Ideal differentiator		$v_{out}(t) = -RC\frac{dv_i}{dt}$

$RC\frac{dv_o}{dt} + v_i = 0$

$\frac{dv_o}{dt} = -\frac{1}{RC}v_i(t)$

$v_o = -RC\frac{dv_i}{dt}$

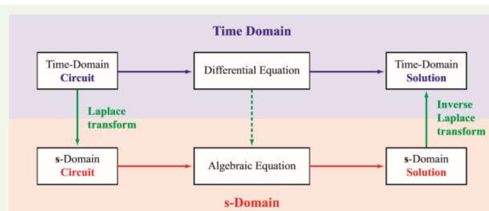


Figure 12-2: The top horizontal sequence involves solving a differential equation entirely in the time domain. The bottom horizontal sequence involves a much easier solution of a linear equation in the s-domain.

$v_s(t) = u(t)$

$\longleftrightarrow U(s) = \frac{1}{s}$

$\downarrow \mathcal{L}[u(t)]$

$$\mathcal{L}[u(t)] = \int_{0-}^{\infty} \underset{1}{u(\tau)}\, e^{-s\tau}\, d\tau$$

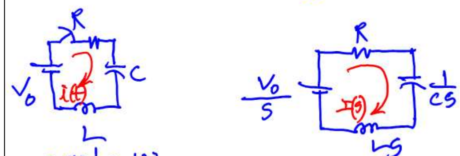
$$= \int_{0-}^{\infty} d\left[\frac{e^{-s\tau}}{-s}\right] = -\frac{1}{s}\left[\underset{0}{e^{-s\infty}} - \underset{1}{e^{s(0)}}\right]$$

$$= \frac{1}{s}$$

$$\boxed{e^{-at}u(t)} \longleftrightarrow \boxed{\frac{1}{s+a}}$$

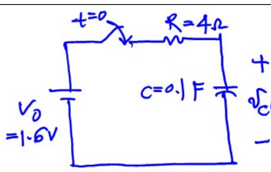
---

t		s
R	$\longleftrightarrow$	R
C	$\longleftrightarrow$	$\frac{1}{Cs}$ (impedance)
L	$\longleftrightarrow$	Ls (impedance)
$u(t)$	$\longleftrightarrow$	$\frac{1}{s}$



initial condition
$v_C(0-)=0 \quad i_L(0-)=0$

$$\frac{V_0}{s} = R\,I(s) + \frac{1}{Cs}I(s) + Ls\,I(s)$$



For $t > 0$

$$V_0 = v_C(t) + 4\left(0.1\frac{dv_C(t)}{dt}\right)$$

$$= v_C(t) + 0.4\frac{dv_C(t)}{dt}$$

$$2.5V_0 = \frac{dv_C(t)}{dt} + 2.5\,v_C(t)$$

$\downarrow$

$$2.5\frac{V_0}{s} = sV_C(s) + 2.5V_C(s)$$

$$= (s+2.5)\,V_C(s)$$

$v_C(t) \longleftrightarrow V_C(s)$

$\frac{dv_C(t)}{dt} \longleftrightarrow sV_C(s) - v_C(0-)$

$$v_C(t) = \left(V_0 - V_0e^{-2.5t}\right)u(t) \xleftarrow{\mathcal{L}^{-1}} V_C(s) = \frac{V_0 \times 2.5}{s[s+2.5]}$$

$$= V_0\left(1 - e^{-2.5t}\right)u(t) \qquad = \frac{V_0}{s} + \frac{-V_0}{s+2.5}$$

---

where $i(t)$ is the current flowing through the loop and $v_C(t)$ is the voltage across C. By invoking the i-v relationship for C, Eq. (12.36) becomes

$$Ri + \left[\frac{1}{C}\int_{0^-}^{t} i\,dt + v_C(0^-)\right] + L\,\frac{di}{dt} = V_o\,u(t), \qquad (12.37)$$

which now contains a single dependent variable, $i(t)$.

Step 2: Define Laplace transform currents and voltages corresponding to the time-domain currents and voltages and then transform the equation to the s-domain

We designate $\mathbf{I(s)}$ as the **s**-domain counterpart of $i(t)$,

$$i(t) \longleftrightarrow \mathbf{I(s)}. \qquad (12.38)$$

To transform Eq. (12.37) to the **s**-domain, we apply the appropriate property or *Laplace transformation* (LT) from Tables 12-1 and 12-2, as follows:

$$R\,i(t) \longleftrightarrow R\,\mathbf{I(s)}$$
(multiplication by constant),

$$\frac{1}{C}\int_{0^-}^{t} i\,dt \longleftrightarrow \frac{1}{C}\,\frac{\mathbf{I(s)}}{\mathbf{s}} \quad \textbf{(time-integral property)},$$

$$v_C(0^-) \longleftrightarrow \frac{v_C(0^-)}{\mathbf{s}} \quad \textbf{(LT of a constant)},$$

$$L\,\frac{di}{dt} \longleftrightarrow L[\mathbf{s}\,\mathbf{I(s)} - i(0^-)]$$
(time derivative property),

$$V_o\,u(t) \longleftrightarrow \frac{V_o}{\mathbf{s}} \quad \textbf{(LT of a constant)}.$$

The opening paragraph of this section stated that the circuit had no stored energy prior to $t = 0$. Hence, $v_C(0^-) = 0$ and $i(0^-) = 0$. Replacing each of the terms in Eq. (12.37) with its **s**-domain counterpart leads to

$$R\mathbf{I} + \frac{\mathbf{I}}{C\mathbf{s}} + L\mathbf{s}\mathbf{I} = \frac{V_o}{\mathbf{s}} \quad \textbf{(s-domain)}. \qquad (12.39)$$

Table 12-1: Properties of the Laplace transform ($f(t) = 0$ for $t < 0^-$).

Property	$f(t)$		$\mathbf{F(s)} = \mathcal{L}[f(t)]$
1. Multiplication by constant	$K\ f(t)$	$\longleftrightarrow$	$K\ \mathbf{F(s)}$
2. Linearity	$K_1\ f_1(t) + K_2\ f_2(t)$	$\longleftrightarrow$	$K_1\ \mathbf{F}_1\mathbf{(s)} + K_2\ \mathbf{F}_2\mathbf{(s)}$
3. Time scaling	$f(at), \quad a > 0$	$\longleftrightarrow$	$\frac{1}{a}\ \mathbf{F}\left(\frac{\mathbf{s}}{a}\right)$
4. Time shift	$f(t-T)\ u(t-T)$	$\longleftrightarrow$	$e^{-T\mathbf{s}}\ \mathbf{F(s)}, \quad T \geq 0$
5. Frequency shift	$e^{-at}\ f(t)$	$\longleftrightarrow$	$\mathbf{F(s}+a\mathbf{)}$
6. Time 1st derivative	$f' = \frac{df}{dt}$	$\longleftrightarrow$	$\mathbf{s}\ \mathbf{F(s)} - f(0^-)$
7. Time 2nd derivative	$f'' = \frac{d^2f}{dt^2}$	$\longleftrightarrow$	$\mathbf{s}^2\mathbf{F(s)} - \mathbf{s}\,f(0^-) - f'(0^-)$
8. Time integral	$\int_{0^-}^{t} f(\tau)\,d\tau$	$\longleftrightarrow$	$\frac{1}{\mathbf{s}}\ \mathbf{F(s)}$
9. Frequency derivative	$t\ f(t)$	$\longleftrightarrow$	$-\frac{d}{d\mathbf{s}}\ \mathbf{F(s)}$
10. Frequency integral	$\frac{f(t)}{t}$	$\longleftrightarrow$	$\int_{\mathbf{s}}^{\infty} \mathbf{F(s')}\ d\mathbf{s'}$

In general, for

$$\boxed{\frac{dx}{dt} + ax = y(t)}$$

$\downarrow \mathcal{L}$

$$sX(s) - x(0-) + aX(s) = Y(s)$$

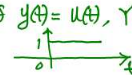
$$X(s)(s+a) = Y(s) + x(0-)$$

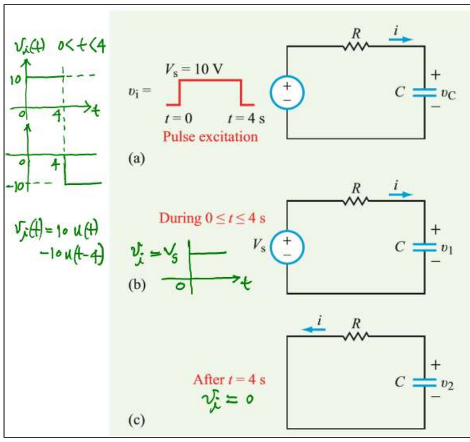
$$X(s) = \frac{Y(s)}{s+a} + \frac{x(0-)}{s+a}$$

$\mathcal{L}^{-1}$ (Inverse Laplace transform)

$$x(t) = \underbrace{\mathcal{L}^{-1}\left(\frac{Y(s)}{s+a}\right)}_{\text{Forced response}} + \underbrace{x(0)e^{-at}}_{\text{Natural response}}$$

Example If $y(t) = u(t)$, $Y(s) = \frac{1}{s}$





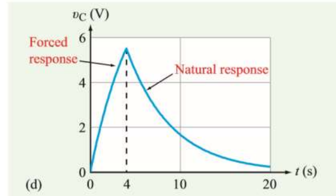


Figure 5-35: RC-circuit response to a 4 s long rectangular pulse.

Solution: According to Example 5-2, a rectangular pulse is equivalent to the sum of two step functions. Thus

$$v_i(t) = V_s[u(t - T_1) - u(t - T_2)],$$

where $u(t - T_1)$ accounts for the rise in level from 0 to 1 at $t = T_1$ and the second term (with negative amplitude) serves to counteract (cancel) the first term after $t = T_2$. For the present problem, $T_1 = 0$, and $T_2 = 4$ s. Hence, the input pulse can be written as

$$v_i(t) = V_s\, u(t) - V_s\, u(t - 4).$$

Response to $V_s\, u(t)$ alone

The response $v_1(t)$ is given by Eq. (5.96) with $v_1(0) = 0$, $v_1(\infty) = V_s$, and $\tau = RC$. Hence,

$$v_1(t) = v_1(\infty) + [v_1(0) - v_1(\infty)]e^{-t/\tau}$$
$$= V_s(1 - e^{-t/\tau}) \qquad (\text{for } t \ge 0).$$

For $V_s = 10$ V and $\tau = RC = 25 \times 10^3 \times 0.2 \times 10^{-3} = 5$ s,

$$v_1(t) = 10(1 - e^{-0.2t})\ \text{V}\ u(t) \qquad (\text{for } t \ge 0).$$

Response to $-V_s\, u(t-4)$ alone

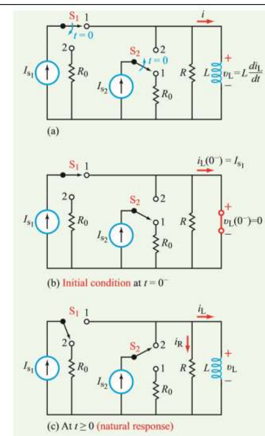
The second step function has an amplitude of $-V_s$ and is delayed in time by 4 s. Upon reversing the polarity of V_s and replacing t with $(t - 4)$, we have

$$v_2(t) = -10[1 - e^{-0.2(t-4)}]\ \text{V} \qquad (\text{for } t \ge 4 \text{ s}).$$

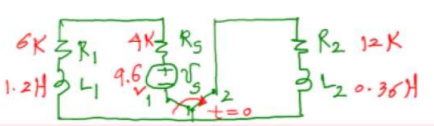
Total response

The total response for $t \ge 0$ therefore is given by

$$v_C(t) = v_1(t) + v_2(t)$$
$$= 10[1 - e^{-0.2t}] \times u(t) - 10[1 - e^{-0.2(t-4)}]\, u(t-4)\ \text{V},$$

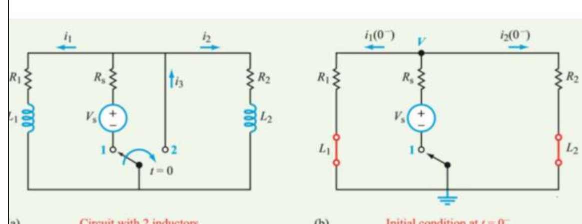


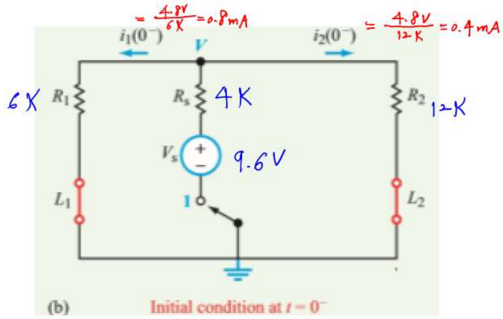
$$I_{s_2} = i_L + \frac{v_L}{R}$$
$$= i_L + \frac{1}{R} L \frac{di_L}{dt}$$



Example 5-13: Circuit with Two RL Branches

After having been in position 1 for a long time, the SPDT switch in **Fig. 5-38(a)** was moved to position 2 at $t = 0$. Determine i_1, i_2, and i_3 for $t \ge 0$, given that $V_s = 9.6$ V, $R_s = 4$ kΩ, $R_1 = 6$ kΩ, $R_2 = 12$ kΩ, $L_1 = 1.2$ H, and $L_2 = 0.36$ H.





KCL at node V : $\frac{V-9.6}{4K} + \frac{V}{6K} + \frac{V}{12K} = 0$

$3(V-9.6) + 2V + V = 0 \Rightarrow 6V = 3\times 9.6$

$V = 4.8V$

to the expression given by Eq. (5.96). Thus, the general form for the current through an inductor in an RL circuit is given by

$$i_L(t) = \left[i_L(\infty) + [i_L(0) - i_L(\infty)]e^{-t/\tau}\right] u(t), \quad (5.107)$$

(switch action at $t = 0$)

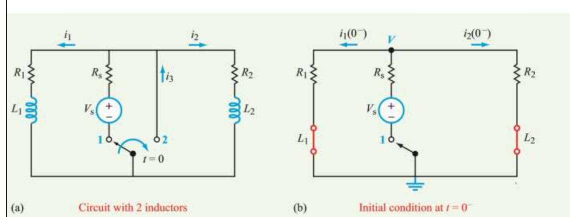
with ***time constant*** $\tau = L/R$. For the specific circuit in **Fig. 5-37(a)**, $i_L(0) = I_{s_1}$ and $i_L(\infty) = I_{s_2}$.

If the sudden change in the circuit configuration happens at $t = T_0$ instead of at $t = 0$, the general expression for $i_L(t)$ becomes

$$i_L(t) = \left\{i_L(\infty) + [i_L(T_0) - i_L(\infty)]e^{-(t-T_0)/\tau}\right\} \cdot u(t - T_0), \quad (5.108)$$

(switch action at $t = T_0$)

where $i_L(T_0)$ is the current at T_0. This expression is the analogue of Eq. (5.98) for the voltage across the capacitor.



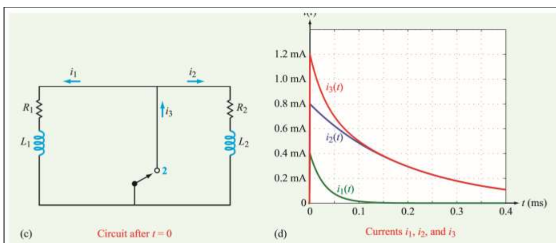


Figure 5-38: Circuit for Example 5-13.

like short circuits, resulting in the equivalent circuit shown in **Fig. 5-38(b)**. Application of KCL to node V gives

$$\frac{V}{R_1} + \frac{V - V_s}{R_s} + \frac{V}{R_2} = 0,$$

whose solution is

$$V = \frac{R_1 R_2 V_s}{R_1 R_2 + R_1 R_s + R_2 R_s} = \frac{6 \times 12 \times 9.6}{6 \times 12 + 6 \times 4 + 12 \times 4} = 4.8 \text{ V}.$$

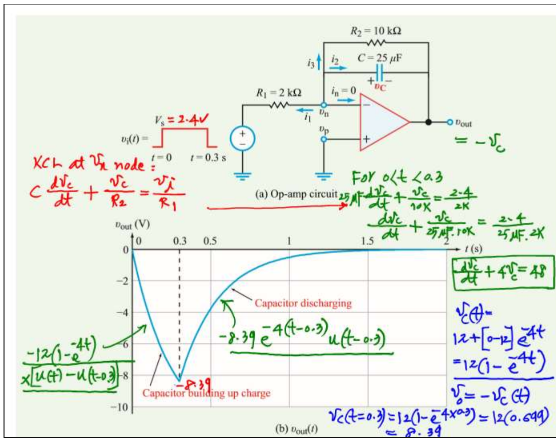
Hence, the initial currents $i_1(0)$ and $i_2(0)$ are given by

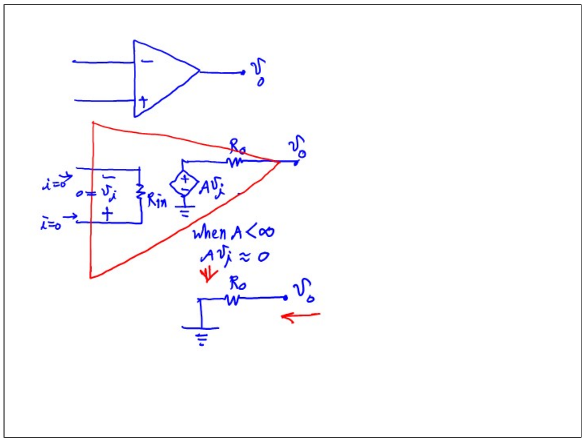
$$i_1(0) = i_1(0^-) = \frac{V}{R_1} = \frac{4.8}{6 \times 10^3} = 0.8 \text{ mA}$$

and

$$i_2(0) = i_2(0^-) = \frac{V}{R_2} = \frac{4.8}{12 \times 10^3} = 0.4 \text{ mA}.$$

The circuit in **Fig. 5-38(c)** represents the natural response circuit condition after $t = 0$. Even though we have two resistors and two inductors in the overall circuit, it can be treated





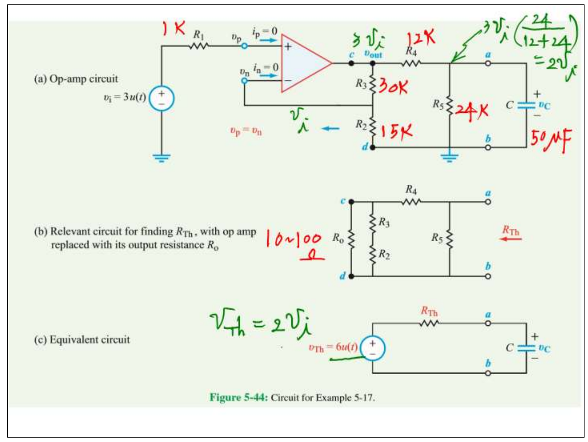
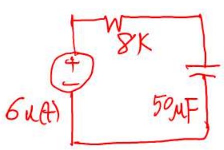


Figure 5-44: Circuit for Example 5-17.

$$R_{TH} = \Big[\underbrace{\{R_0 \,\|\, (R_2 + R_3)\}}_{\text{small}} + R_4\Big] \,\|\, R_5$$

$$\approx R_4$$

$$\approx R_4 \,\|\, R_5 = \frac{\cancel{12K} \times \cancel{24K}}{\cancel{12K} + \cancel{24K}} = \underline{8K\Omega}$$



$$\frac{dv_C(t)}{dt} + \frac{v_C}{8K \times 50\mu} = 6, \quad t > 0$$

$$v_C(t) = 6 + (0 - 6)e^{-2.5t}$$

$$= \underline{6(1 - e^{-2.5t})\, u(t)}$$