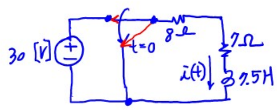
EE101 Lecture 13, Feb 19, 2019

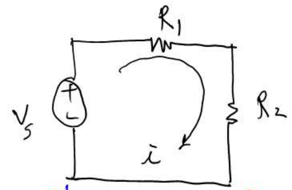
Quiz 7 on Feb 26 based on HW 7.

[1] Prob. 6.1
[2] 6.2
[3] 6.3
[4] 6.4
[5] 6.5
[6] 6.7

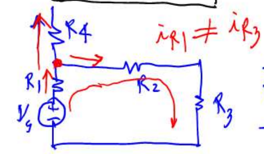
[7] Find $i(t)$, $t>0$ for the circuit below



Quiz5 Average = 7.08, σ = 2.34



$$V_{R_2} = \left(\frac{V_s}{R_1+R_2}\right) R_2 = \left(\frac{R_2}{R_1+R_2}\right) V_s$$



$i_{R_1} \neq i_{R_3}$

$$V_{R_3} = V_s\left(\cancel{\frac{R_2}{R_1+R_2+R_3}}\right)$$

RC and RL First-Order Circuits

Contents

Objectives

Learn to:

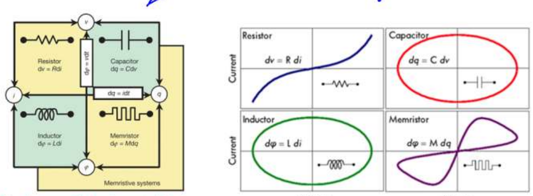
- Use mathematical functions to describe several types of nonperiodic waveforms.
- Define the electrical properties of a capacitor, including its i-υ relationship and energy equation.
- Combine multiple capacitors when connected in series or in parallel.
- Define the electrical properties of an inductor, including its i-υ relationship and energy equation.
- Combine multiple inductors when connected in series or in parallel.

Capacitors (C) and inductors (L) are ***energy storage devices***, in contrast with resistors, which are energy dissipation devices. This chapter examines the behavior of ***RC and RL circuits***, to be followed in Chapter 6 with an examination of RLC circuits.

- Analyze the transient responses of RC and RL circuits.
- Design RC op-amp circuits to perform differentiation and integration and related operations.
- Apply Multisim to analyze RC and RL circuits.

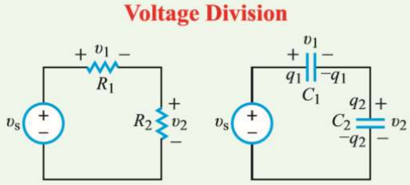
Capacitor $dq = C\,dv$

Linear case $q = Cv$



Memristor as 4th fundamental device

Voltage Division



(a) $\upsilon_1 = \left(\frac{R_1}{R_1+R_2}\right)\upsilon_s$ (b) $\upsilon_1 = \left(\frac{C_2}{C_1+C_2}\right)\upsilon_s$

$\upsilon_2 = \left(\frac{R_2}{R_1+R_2}\right)\upsilon_s$ $\upsilon_2 = \left(\frac{C_1}{C_1+C_2}\right)\upsilon_s$

Figure 5-19: Voltage-division rules for (a) in-series resistors and (b) in-series capacitors.

same as resistor connection

Combining In-Series Inductors

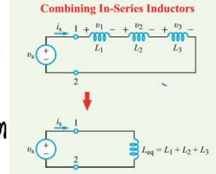
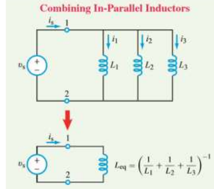
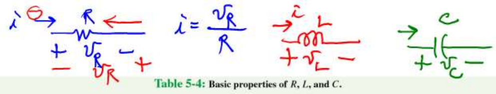


Figure 5-24: Inductors in series.

Combining In-Parallel Inductors

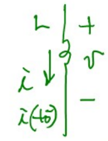


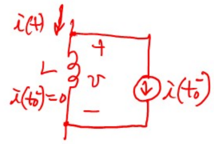


$i = \frac{v_R}{R}$

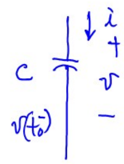
Table 5-4: Basic properties of R, L, and C.

Property	R	L	C
i–v relation	$i = \frac{v}{R}$	$i = \frac{1}{L}\int_{t_0}^{t} v\, dt' + i(t_0)$	$i = C\frac{dv}{dt}$
v-i relation	$v = iR$	$v = L\frac{di}{dt}$	$v = \frac{1}{C}\int_{t_0}^{t} i\, dt' + v(t_0)$
p (power transfer in)	$p = i^2 R$	$p = Li\frac{di}{dt}$	$p = Cv\frac{dv}{dt}$
w (stored energy)	0	$w = \frac{1}{2}Li^2$	$w = \frac{1}{2}Cv^2$
Series combination	$R_{eq} = R_1 + R_2$	$L_{eq} = L_1 + L_2$	$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$
Parallel combination	$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$	$\frac{1}{L_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$	$C_{eq} = C_1 + C_2$
dc behavior	no change	short circuit	open circuit
Can v change instantaneously?	yes	yes	no
Can i change instantaneously?	yes	no	yes

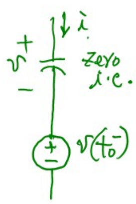




$$i(t) = \frac{1}{L}\int_{t_0}^{t} v(\tau)\, d\tau + \underline{i(t_0^-)}$$



$$v = \frac{1}{C}\int_{t_0}^{t} i(\tau)\, d\tau + v(t_0^-)$$



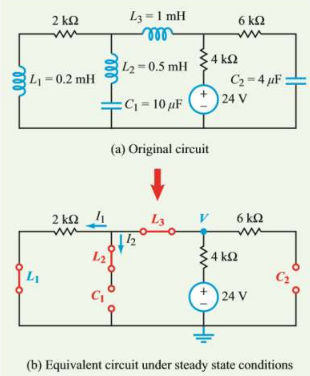
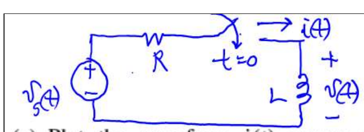


Figure 5-26: Under steady-state dc conditions, capacitors act like open circuits, and inductors act like short circuits.

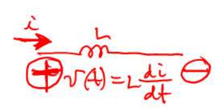


(a) Plot the waveform $i(t)$ versus t and determine the locations of its first maximum, first minimum, and their corresponding amplitudes.

(b) given that $L = 50$ mH, obtain an expression for $v(t)$ across the inductor and plot its waveform.

(c) Generate a plot of the power $p(t)$ delivered to the inductor.

$$v(t) = L\frac{d\,i(t)}{dt}$$



Upon closing the switch at $t = 0$ in the circuit of **Fig. 5-23(a)**, the voltage source generates a current waveform through the circuit given by

$$i(t) = 10e^{-0.8t}\sin(\pi t/2)\ \text{A}, \qquad (\text{for } t \geq 0).$$

$$\frac{d}{dt}i(t) = 10(-0.8)e^{-0.8t}\sin(\pi t/2) + 10\,e^{-0.8t}\,\pi/2\,\cos(\pi t/2)$$

$$v(t) = 50\times10^{-3}\times\left[-8\,e^{-0.8t}\sin\pi t/2 + 5\pi\,e^{-0.8t}\cos(\pi t/2)\right]$$

$$p(t) = v(t)\, i(t)$$
$$= 50\times10^{-3}\left[-8e^{-0.8t}\sin(\pi t/2) + 5\pi e^{-0.8t}\cos(\pi t/2)\right]$$
$$\times\ 10e^{-0.8t}\sin(\pi t/2)$$
$$= -4e^{-1.6t}\sin^2(\pi t/2) + 2.5e^{-1.6t}\sin(\pi t/2)\cos(\pi t/2)$$
$$= -2e^{-1.6t}(1-\cos\pi t) + 1.25e^{-1.6t}\sin(\pi t)$$

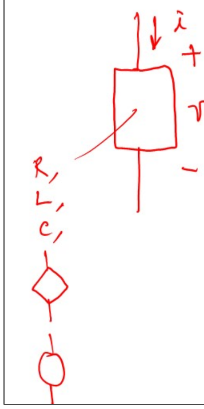
$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

$$= -2e^{-1.6t} + e^{-1.6t}(2\cos\pi t + 1.25\sin\pi t)$$

at $t=0$ $p(t=0) = -2 + (2\times1 + 1.25\times0)$ $= 0$ W (makes sense)

as $t\to\infty$ $p(t)\to 0$



$p = v\, i$

L: $v = L\frac{di}{dt}$

i

$p(t) = v\, i$
$= L\, i \cdot \frac{di}{dt}$

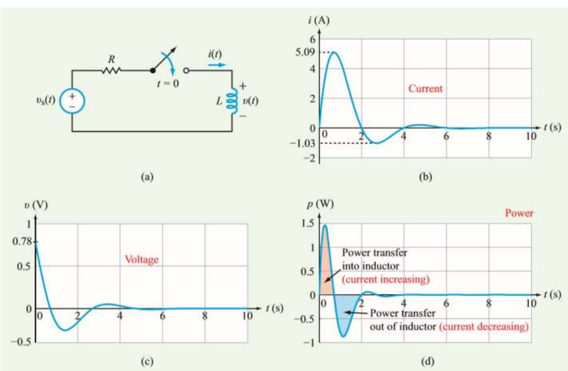


Figure 5-23: Circuit for Example 5-7.

DC source

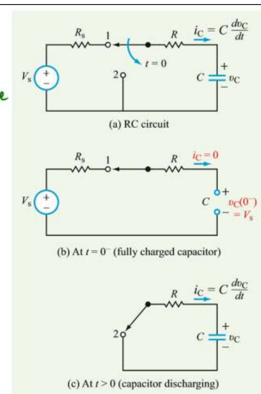


Figure 5-28: RC circuit with an initially charged capacitor that starts to discharge its energy after $t = 0$.

for $t<0$ switch at position 1

In dc, $C\frac{dv_c}{dt} = 0$ open circuit

⇒ no voltage drop across R

thus $v_c(0^-) = V_s$

$t>0$

$$\boxed{C\frac{dv_c}{dt} = -\frac{v_c}{R} \qquad v_c(0^-) = V_s}$$

$$RC\,\frac{dv_C}{dt} + v_C = 0. \tag{5.69}$$

Upon dividing both terms by RC, Eq. (5.69) takes the form

$$\frac{dv_C}{dt} + av_C = 0 \qquad \text{(source-free)}, \tag{5.70}$$

$v_c(0^-) = V_s$ (i.c.)

where

$$\boxed{\frac{dx}{dt} + ax = 0}$$

$$a = \frac{1}{RC}. \tag{5.71}$$

► The solution of the source-free equation is called the *natural response* (discharging condition) of the circuit. ◄

The standard procedure for solving Eq. (5.70) starts by replacing t with dummy variable t' and multiplying both sides by $e^{at'}$,

$$\frac{dv_C}{dt'}\,e^{at'} + av_Ce^{at'} = 0. \tag{5.72}$$

Next, we recognize that the sum of the two terms on the left-hand side is equal to the expansion of the differential of $(v_Ce^{at'})$,

$$\frac{d}{dt'}(v_Ce^{at'}) = \frac{dv_C}{dt'}\,e^{at'} + av_Ce^{at'}. \tag{5.73}$$

Hence, Eq. (5.72) becomes

$$\frac{d}{dt'}(v_Ce^{at'}) = 0. \tag{5.74}$$

Integrating both sides, we have

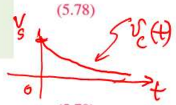
$$\int_0^t \frac{d}{dt'}(v_Ce^{at'})\,dt' = 0, \tag{5.75}$$

where we have chosen the lower limit to be $t' = 0$ (because we are given specific information on the state of the circuit at that point in time). Performing the integration gives

$$v_Ce^{at'}\Big|_0^t = 0$$

$$v_C(t) = v_C(0)\, e^{-t/\tau}, \qquad (5.78)$$

(natural response discharging),

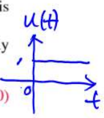


with

$$\tau = RC \qquad \text{(s)}, \qquad (5.79)$$

where τ is called the ***time constant*** of the circuit, and it is measured in seconds (s).

In view of the initial condition given by Eq. (5.67), namely $v_C(0) = V_s$, the expression for $v_C(t)$ becomes



$$v_C(t) = V_s e^{-t/\tau}\, u(t), \qquad (5.80)$$

where we inserted the unit step function $u(t)$ as a multiplication factor as a substitute for "for $t \geq 0$." The plot shown in **Fig. 5-29(a)** indicates that in response to the switch action, $v_C(t)$ decays exponentially with time from V_s at $t = 0$ down to its final value of zero as $t \to \infty$. The decay rate is dictated by the time constant τ. At $t = \tau$,

$$v_C(t = \tau) = V_s e^{-1} = 0.37 V_s, \qquad (5.81)$$

For $t > 0$

$i_C = C \frac{dv_C}{dt}$

$= C \frac{d}{dt}\left(V_s e^{-t/\tau}\right)$

$= V_s \cancel{C} \left(-\frac{1}{\tau}\right) V_s\, e^{-t/\tau}$ $\quad R\cancel{C}$

$= (-)\frac{V_s}{R}\, e^{-t/\tau}$

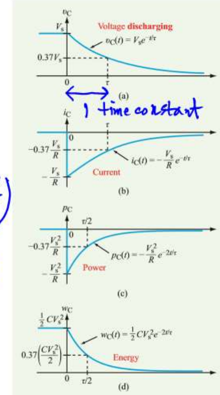


Figure 5-29: Response of the RC circuit in **Fig. 5-28(a)** to moving the SPDT switch to terminal 2.

$e^{-t/\tau}$, $\tau = RC$

$t = \tau$

$e^{-1} = 0.37$

Series RC Circuit Solution

1: If switch action is at $t = 0$, analyze circuit at $t = 0^-$ to determine initial conditions $v_C(0^-)$ and $i_C(0^-)$. Use this information to determine $v_C(0)$ and $i_C(0)$, at t immediately after the switch action. Remember that the voltage across a capacitor cannot change instantaneously (between $t = 0^-$ and $t = 0$), but the current can.

2: Analyze the circuit to determine $v_C(\infty)$, the voltage across the capacitor long after the switch action.

3: Determine the time constant $\tau = RC$.

4: Incorporate the information obtained in the previous three steps in Eq. (5.96):

$$v_C(t) = \left\{v_C(\infty) + [v_C(0) - v_C(\infty)]e^{-t/\tau}\right\} u(t).$$

5: If the switch action is at $t = T_0$ instead of $t = 0$, replace 0 with T_0 and use Eq. (5.98):

$$v_C(t) = \left\{v_C(\infty) + [v_C(T_0) - v_C(\infty)] \cdot e^{-(t-T_0)/\tau}\right\} \cdot u(t - T_0).$$

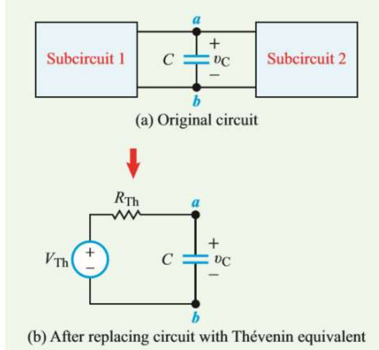


Figure 5-31: Replacing a resistive circuit with its Thévenin equivalent as seen by capacitor C.

Thévenin Approach to RC Response

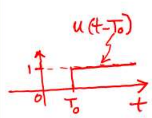
Step 1: If the circuit includes a single switch action (open, close, or move between two terminals) at $t = T_0$, analyze the circuit at $t = T_0^-$ (just before the switch action) to determine $v_C(T_0^-)$. When so doing, the capacitor should be replaced with an open circuit. Then set $v_C(T_0) = v_C(T_0^-)$, where $v_C(T_0)$ is the voltage across the capacitor *after* the switch action.

Step 2: For the circuit configuration at $t \geq T_0$ (after the switch action), obtain the Thévenin equivalent circuit as "seen" by the capacitor. **Figure 5-31(a)** depicts a general circuit (composed of possibly two subcircuits) connected to a capacitor C. After removing (temporarily) the capacitor and calculating V_{Th} and R_{Th} of the equivalent Thévenin circuit at terminals (a, b), reinstate the capacitor as in **Fig. 5-31(b)**.

Step 3: The capacitor's voltage response is then given by

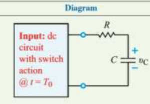
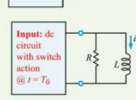
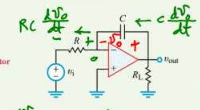
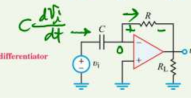
$$v_C(t) = \left\{v_C(\infty) + [v_C(T_0) - v_C(\infty)]e^{-(t-T_0)/\tau}\right\} \cdot u(t - T_0),$$

with $v_C(\infty) = V_{Th}$, $v_C(T_0)$ as obtained in step 1, and $\tau = R_{Th}C$.



Step 4: If the circuit undergoes multiple switch actions, repeat the procedure for each time segment and use the property that the voltage across a capacitor cannot change instantaneously to match the responses at the boundaries between adjacent time segments.

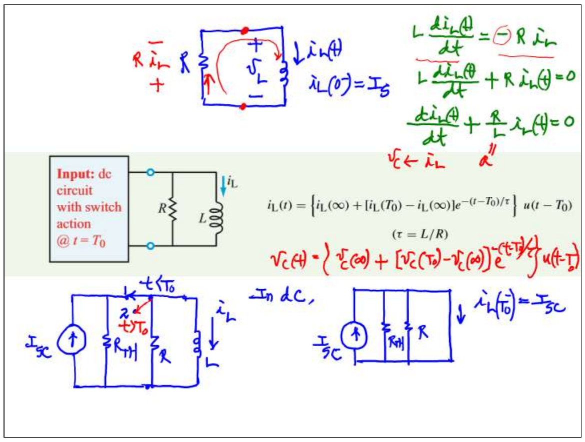
Table 5-5: Response forms of basic first-order circuits.

Circuit	Diagram	Response
RC		$v_C(t) = \left\{v_C(\infty) + [v_C(T_0) - v_C(\infty)]e^{-(t-T_0)/\tau}\right\} u(t - T_0)$ $(\tau = RC)$
RL		$i_L(t) = \left\{i_L(\infty) + [i_L(T_0) - i_L(\infty)]e^{-(t-T_0)/\tau}\right\} u(t - T_0)$ $(\tau = L/R)$
Ideal integrator		$v_{out}(t) = -\frac{1}{RC}\int_{t_0}^{t} v_i\, dt' + v_{out}(t_0)$
Ideal differentiator		$v_{out}(t) = -RC\frac{dv_i}{dt}$

$RC\frac{dv_0}{dt} + v_i = 0$

$\frac{dv_0}{dt} = -\frac{1}{RC} v_i(t)$

$v_0 = -RC\frac{dv_i}{dt}$



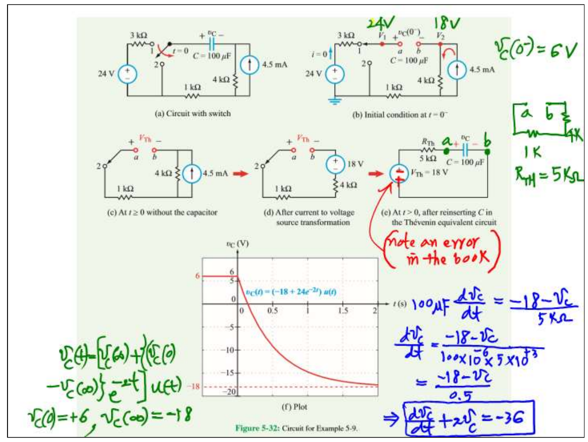


Figure 5-32: Circuit for Example 5-9.

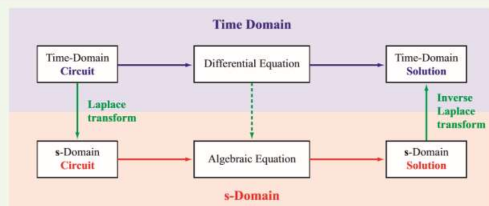


Figure 12-2: The top horizontal sequence involves solving a differential equation entirely in the time domain. The bottom horizontal sequence involves a much easier solution of a linear equation in the **s**-domain.

Solution Procedure: Laplace Transform

Step 1: The circuit is transformed to the Laplace domain—also known as the **s**-domain.

Step 2: In the **s**-domain, application of KVL and KCL yields a set of algebraic equations.

Step 3: The equations are solved for the variable of interest.

Step 4: The **s**-domain solution is transformed back to the time domain.

The uniqueness property can be expressed in symbolic form by

$$f(t) \longleftrightarrow \mathbf{F}(\mathbf{s}). \qquad (12.12a)$$

The two-way arrow is a short-hand notation for the combination of the two statements

$$\mathcal{L}[f(t)] = \mathbf{F}(\mathbf{s}), \qquad \mathcal{L}^{-1}[\mathbf{F}(\mathbf{s})] = f(t). \qquad (12.12b)$$

The first statement asserts that $\mathbf{F}(\mathbf{s})$ is the Laplace transform of $f(t)$, and the second one asserts that the ***inverse Laplace transform*** ($\mathcal{L}^{-1}[\;]$) of $\mathbf{F}(\mathbf{s})$ is $f(t)$.

12-2.1 Definition of the Laplace Transform

The symbol $\mathcal{L}[f(t)]$ is a short-hand notation for "the Laplace transform of function $f(t)$." Usually denoted $\mathbf{F(s)}$, the Laplace transform is defined by

$$\mathbf{F(s)} = \mathcal{L}[f(t)] = \int_{0^-}^{\infty} f(t)\, e^{-\mathbf{s}t}\, dt, \qquad (12.10)$$

where $\mathbf{s}$ is a complex variable with a real part σ and an imaginary part ω:

$$\mathbf{s} = \sigma + j\omega. \qquad (12.11)$$

Given that the exponent $\mathbf{s}t$ has to be dimensionless, $\mathbf{s}$ has the unit of inverse second, which is the same as Hz or rad/s. Moreover, since $\mathbf{s}$ is a complex quantity, it is often termed ***complex frequency***.

$v_s(t) = u(t)$

$\longleftrightarrow U(s) = \frac{1}{S}$

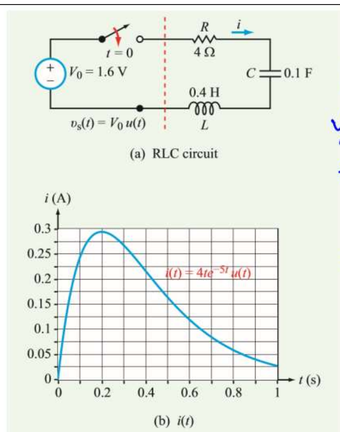
$\downarrow \mathcal{L}[u(t)]$

$$\mathcal{L}[u(t)] = \int_{0^-}^{\infty} \underset{1}{u(\tau)}\, e^{-s\tau} d\tau$$

$$= \int_{0^-}^{\infty} d\left[\frac{e^{-s\tau}}{-s}\right] = -\frac{1}{s}\left[\underset{0}{e^{-s\infty}} - \underset{1}{e^{s(0)}}\right]$$

$$= \frac{1}{s}$$

$$\boxed{e^{-at}u(t)} \longleftrightarrow \boxed{\frac{1}{s+a}}$$



(a) RLC circuit

(b) $i(t)$

For $t > 0$

$$V_0 = Ri(t) + \frac{1}{C}\int_0^t i(t)dt + v_c(0^-) + L\frac{di(t)}{dt}$$

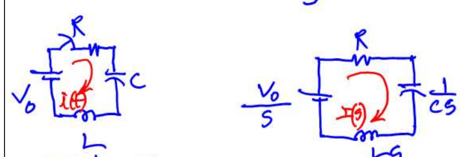
$t \qquad s$

$R \longleftrightarrow R$

$C \longleftrightarrow \frac{1}{Cs}$ (impedance)

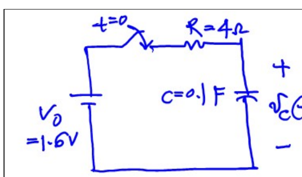
$L \longleftrightarrow Ls$ (impedance)

$u(t) \longleftrightarrow \frac{1}{s}$



initial condition
$v_c(0^-) = 0 \quad i_L(0^-) = 0$

$$\frac{V_0}{s} = RI(s) + \frac{1}{Cs}I(s) + LsI(s)$$



For $t > 0$

$$V_0 = v_c(t) + 4\left(0.1\frac{dv_c(t)}{dt}\right)$$
$$= v_c(t) + 0.4\frac{dv_c(t)}{dt}$$
$$2.5V_0 = \frac{dv_c(t)}{dt} + 2.5v_c(t)$$

$\downarrow$

$v_c(t) \longleftrightarrow V_c(s)$

$\frac{dv_c(t)}{dt} \longleftrightarrow sV_c(s) - v_c(0^-)$

$$2.5\frac{V_0}{s} = sV_c(s) + 2.5V_c(s)$$
$$= (s+2.5)V_c(s)$$

$$V_c(s) = \frac{V_0 \times 2.5}{s[s+2.5]}$$
$$= \frac{V_0}{s} + \frac{-V_0}{s+2.5}$$

$$v_c(t) = V_0 + (-V_0)e^{-2.5t}$$
$$= V_0\left[1 - e^{-2.5t}\right]u(t)$$

where $i(t)$ is the current flowing through the loop and $\upsilon_C(t)$ is the voltage across C. By invoking the i-υ relationship for C, Eq. (12.36) becomes

$$Ri + \left[\frac{1}{C}\int_{0^-}^{t} i\,dt + \upsilon_C(0^-)\right] + L\,\frac{di}{dt} = V_o\,u(t), \qquad (12.37)$$

which now contains a single dependent variable, $i(t)$.

Step 2: Define Laplace transform currents and voltages corresponding to the time-domain currents and voltages and then transform the equation to the s-domain

We designate $\mathbf{I}(\mathbf{s})$ as the $\mathbf{s}$-domain counterpart of $i(t)$,

$$i(t) \leftrightarrow \mathbf{I}(\mathbf{s}). \qquad (12.38)$$

To transform Eq. (12.37) to the $\mathbf{s}$-domain, we apply the appropriate property or ***Laplace transformation*** (LT) from Tables 12-1 and 12-2, as follows:

$$R\,i(t) \leftrightarrow R\,\mathbf{I}(\mathbf{s})$$

(multiplication by constant),

$$\frac{1}{C}\int_{0^-}^{t} i\,dt \leftrightarrow \frac{1}{C}\,\frac{\mathbf{I}(\mathbf{s})}{\mathbf{s}} \quad \textbf{(time-integral property)},$$

$$\upsilon_C(0^-) \leftrightarrow \frac{\upsilon_C(0^-)}{\mathbf{s}} \quad \textbf{(LT of a constant)},$$

$$L\,\frac{di}{dt} \leftrightarrow L[\mathbf{s}\,\mathbf{I}(\mathbf{s}) - i(0^-)]$$

(time derivative property),

$$V_o\,u(t) \leftrightarrow \frac{V_o}{\mathbf{s}} \quad \textbf{(LT of a constant)}.$$

The opening paragraph of this section stated that the circuit had no stored energy prior to $t = 0$. Hence, $\upsilon_C(0^-) = 0$ and $i(0^-) = 0$. Replacing each of the terms in Eq. (12.37) with its $\mathbf{s}$-domain counterpart leads to

$$R\mathbf{I} + \frac{\mathbf{I}}{C\mathbf{s}} + L\mathbf{s}\mathbf{I} = \frac{V_o}{\mathbf{s}} \quad \textbf{(s-domain)}. \qquad (12.39)$$

Table 12-1: Properties of the Laplace transform ($f(t) = 0$ for $t < 0^-$).

Property	$f(t)$		$\mathbf{F}(\mathbf{s}) = \mathcal{L}[f(t)]$
1. Multiplication by constant	$K\,f(t)$	$\leftrightarrow$	$K\,\mathbf{F}(\mathbf{s})$
2. Linearity	$K_1\,f_1(t) + K_2\,f_2(t)$	$\leftrightarrow$	$K_1\,\mathbf{F}_1(\mathbf{s}) + K_2\,\mathbf{F}_2(\mathbf{s})$
3. Time scaling	$f(at), \quad a > 0$	$\leftrightarrow$	$\frac{1}{a}\,\mathbf{F}\left(\frac{\mathbf{s}}{a}\right)$
4. Time shift	$f(t-T)\,u(t-T)$	$\leftrightarrow$	$e^{-T\mathbf{s}}\,\mathbf{F}(\mathbf{s}), \quad T \geq 0$
5. Frequency shift	$e^{-at}\,f(t)$	$\leftrightarrow$	$\mathbf{F}(\mathbf{s}+a)$
6. Time 1st derivative	$f' = \frac{df}{dt}$	$\leftrightarrow$	$\mathbf{s}\,\mathbf{F}(\mathbf{s}) - f(0^-)$
7. Time 2nd derivative	$f'' = \frac{d^2f}{dt^2}$	$\leftrightarrow$	$\mathbf{s}^2\mathbf{F}(\mathbf{s}) - \mathbf{s}\,f(0^-) - f'(0^-)$
8. Time integral	$\int_{0^-}^{t} f(\tau)\,d\tau$	$\leftrightarrow$	$\frac{1}{\mathbf{s}}\,\mathbf{F}(\mathbf{s})$
9. Frequency derivative	$t\,f(t)$	$\leftrightarrow$	$-\frac{d}{d\mathbf{s}}\,\mathbf{F}(\mathbf{s})$
10. Frequency integral	$\frac{f(t)}{t}$	$\leftrightarrow$	$\int_{\mathbf{s}}^{\infty} \mathbf{F}(\mathbf{s}')\,d\mathbf{s}'$